Noncausal Models

- Causality is not natural in images
- Replace causal prediction with noncausal prediction

1-D noncausal

$p=2$

causal

$$E[Y_n \mid Y_i \ i \neq n]$$

Assume: Y_n is a stationary Gaussian random process.

Defn: Y is call a Gaussian Markov Random Field (GMRF) of order p if

$$E[Y_n \mid Y_i \ i \neq n] = E[Y_n \mid Y_{n-p} \ldots Y_{n-1}, Y_{n+1} \ldots Y_n$$

Lecture 8

- Then define

$$X_n = Y_n - \sum_{\substack{i=-p \\ i\neq 0}}^{p} g_i Y_{n-i} \qquad (g_i = g_{-i})$$

prediction error

noncausal, optimal prediction filter

- Since g is the optimal predictor:

$$E[X_n Y_{n+K}] = 0 \quad K \neq 0$$

$$E[X_n Y_n] = E\Big[X_n \Big(X_n + \sum_{\substack{i=-p \\ i\neq 0}}^{p} g_i Y_{n-i}\Big)\Big]$$

$$= E[X_n^2] = \sigma_{NC}^2$$

$$E[X_n Y_{n+K}] = \sigma_{NC}^2 \delta_K$$

- For $K \neq 0$

$$E[X_n Y_{n+K}] = 0$$

$$= E\Big[X_n \Big(X_{n+K} + \sum_{\substack{i=-p \\ i\neq 0}}^{p} g_i Y_{n+K-i}\Big)\Big]$$

$$= R_X(K) + \underbrace{g_{-K}}_{= g_K} \underbrace{E[X_n Y_n]}_{= \sigma_{NC}^2}$$

$$R_X(K) = \sigma_{NC}^2 (\delta_K - g_K)$$

$$S_x(\omega) = \sigma_{NC}^2 (1 - G(\omega)) = \sigma_{NC}^2 |1 - G(\omega)|$$

$$S_y(\omega) = \frac{1}{|1-G(\omega)|^2} S_x(\omega)$$

$$\boxed{S_y(\omega) = \frac{\sigma_{NC}^2}{1 - G(\omega)}}$$

__Properties__

P1) The prediction errors are not white!

$$S_x(\omega) = \sigma_{NC}^2 (1 - G(\omega))$$

Assume that Y is AR order P

$$S_y(\omega) = \frac{\sigma_c^2}{|1 - H(\omega)|^2}$$

causal predictor

$$\stackrel{?}{=} \frac{\sigma_{NC}^2}{1 - G(\omega)}$$

order?

$$\sigma_c^2 (1 - G(\omega)) = \sigma_{NC}^2 |1 - H(\omega)|^2$$

$$\Downarrow \text{ IDTFT}$$

$$\sigma_c^2 (\delta_n - g_n) = \sigma_{NC}^2 \left((\delta_n - h_n) * (\delta_n - h_{-n}) \right)$$

$\Longrightarrow$ g_n is order P
always FIR

$n = 0$

$$\sigma_c^2 = \sigma_{NC}^2 \left(1 + \sum_{n=1}^{P} h_n^2 \right)$$

$$\sigma_{NC}^2 = \frac{\sigma_c^2}{1 + \sum h_n^2}$$

P2) 1-D $\overset{\text{Gaussian}}{\text{AR}}$ (order P) $\Longrightarrow$ 1-D GMRF (order P)

$$\sigma_{NC}^2 = \frac{\sigma_c^2}{1 + \sum_{n=1}^{P} h_n^2}$$

$$\delta_n - g_n = \frac{(1 - h_n) * (1 - h_{-n})}{1 + \sum_{n=1}^{P} h_n^2}$$

P3)

1-D Gau. AR order P $\Longleftrightarrow$ 1-D GMRF order P

(not true in 2-D)

Equivalent to causal whitening filters used for Weiner filtering

Lecture 9

Example

$$h_n = \begin{cases} \rho & n = -1 \\ 0 & o.w. \end{cases}$$

$$\delta_n - h_n \Rightarrow -\rho \quad 1 \qquad \sigma_c^2 = 1$$

$$(\delta_n - h_n) * (\delta_n - h_{-n}) = \quad -\rho \quad 1+\rho^2 \quad -\rho$$

$$= \frac{\sigma_c^2}{\sigma_{ne}^2} (\delta_n - g_n)$$

$$\sigma_{ne}^2 = \frac{\sigma_c^2}{1+\rho^2}$$

$$g_n = \frac{\rho}{1+\rho^2} \quad 0 \quad \frac{\rho}{1+\rho^2}$$

2-D GMRF

Let Y_s be AR

$$\sigma_c^2 (\delta_s - g_s) = \sigma_{NC}^2 \left[(\delta_s - h_s) * (\delta_s - h_{-s}) \right]$$

g_s is FIR, symmetric
what is its support

Example

$$\begin{matrix} \circ & \times & \circ \\ \circ & \circ & \circ \end{matrix} \quad = \delta_s - h_s$$

$$\begin{matrix} & \circ & \circ & \circ & \circ \\ \circ & \circ & \times & \circ & \circ \\ \circ & \circ & \circ & \circ & \end{matrix} \quad = \delta_s - g_s$$

P1) 2-D Gaussian AR $\Rightarrow$ 2-D GMRF (with proper prediction window

$$\sigma_{NC}^2 = \frac{\sigma_c^2}{1 + \sum_{s>0} h_s^2}$$

$$\delta_s - g_s = \frac{(1 - h_s) * (1 - h_s)}{1 + \sum_{s>0} h_s^2}$$

P2) 2-D GMRF $\not\Rightarrow$ 2-D Gaussian AR

$\sqrt{|1-G(\omega)|}$ is not alway FIR!

<u>Example</u>

$$\delta_s - g_s = \begin{array}{ccc} 0 & -\rho & 0 \\ -\rho & 1 & -\rho \\ 0 & -\rho & 0 \end{array}$$

There is no FIR filter $\delta_n - h_n$ so that

$$(\delta_s - h_s) * (\delta_s - h_{-s}) = \delta_s - g_s$$

Distribution of GMRF (important)

$\{Y_n\}_{n=0}^{N}$ GMRF

$$p(y) = \frac{1}{(\sqrt{2\pi}\,\sigma_{NC})^N} |B|^{1/2} \exp\left\{ -\frac{1}{2\sigma_{NC}^2} Y^T B Y \right\}$$

$$B_{j,k} \equiv \begin{cases} 1 & j = k \\ -g_{j-k} & \text{o.w.} \end{cases}$$

$|B| \neq 1$ Hard to Compute