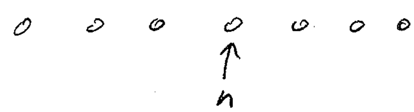
# Markov Random Fields

Objective - Use the noncausal conditional expectation to model a signal / image.



$$p_n(x_n \mid x_i \; i \neq n) = p_n(x_n \mid x_{\partial n})$$

$x_{\partial n}$ - neighbors of $x_n$

How do we write down the distribution for $x$?

$$p(x) \cancel{=} \prod_{n=1}^{N} p_n(x_n \mid x_{\partial n}) \quad ? \quad NO!$$

## Neighborhoods

Define

$$\partial s = \{ r : r \text{ is a neighbor of } s \}$$

Example

4pt neighborhood $\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \\ & \circ & \end{array}$

8pt neighborhood $\begin{array}{ccc} \circ & \circ & \circ \\ \circ & \times & \circ \\ \circ & \circ & \circ \end{array}$

- Every neighborhood must have the following properties

$$\left.\begin{array}{l} 1) \quad s \notin \partial s \\ \\ 2) \quad s \in \partial r \iff r \in \partial s \end{array}\right\} \text{ important}$$

Notation:

$$X_{\partial s} \triangleq \{ X_i : i \in \partial s \}$$

Example Is $\begin{array}{cc} \bullet & \bullet \\ \bullet & \times \end{array}$ a neighborhood system?

$\begin{array}{cc} \circ & \circ \\ & \times \end{array}$

NO! $\begin{array}{cc} a & b \\ c & d \end{array}$ $\underbrace{a \in \partial d \text{ but } d \notin \partial a}_{\text{not allowed}}$

$\Rightarrow$ must be noncausal

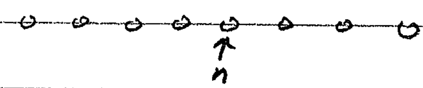
# Lecture 15

- Markov Random Field (MRF)

Defn: $X$ is an MRF with neighborhood system $\partial s$ if

$$p(x_s \mid x_i \; i \neq s) = p(x_s \mid x_{\partial s})$$

Example 1-D



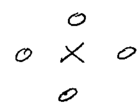
$$p(x_n \mid x_i \; i \neq n) = p(x_n \mid x_{n-1}, x_{n+1})$$

$$\partial n = \{ n-1, n+1 \}$$

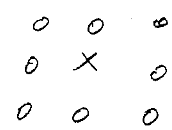
# Gibbs Distribution

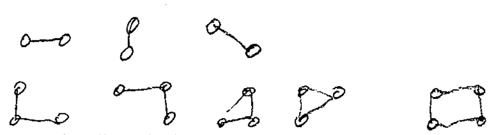
**Defn** $c \subseteq S$ is a clique on the lattice $S$ with neighborhood system $\partial S$ if $\forall a, b \in c$ $a \in \partial b$

**Example**

1) 4pt neighbors 

   cliques 

2) 8pt neighbors 

   cliques 

$$\mathcal{C} \triangleq \{ c : c \text{ is a clique} \}$$

**Defn** $p(x)$ is a Gibbs distribution if it can be written in the form

$$p(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{T} \sum_{c \in \mathcal{C}} V_c(x_c) \right\}$$

Defn $X$ is strictly positive if

for all $x$ $p(x) > 0$.

The Hammersley Clifford theorem:

Let $X$ be a strictly positive random field. Then $X$ is an MRF $\Longleftrightarrow$ $p(x)$ is a Gibbs distribution

proof

$(\Leftarrow)$

$$p(x_s \mid x_i \; i \neq s) = \frac{p(x)}{p(x_i \; i \neq s)}$$

$$= \frac{\frac{1}{Z} \exp\left\{ - \sum_{c \in C} V_c(x_c) \right\}}{\sum_{x_s} \frac{1}{Z} \exp\left\{ - \sum_{c \in C} V_c(x_c) \right\}}$$

Define $C_s = \{ c : c \in C \text{ and } s \in c \}$

$\uparrow$ cliques with the point $s$

$\bar{C}_s = C - C_s$

$\uparrow$ cliques without the point $s$

$$p(x_s \mid x_i \; i \neq s) =$$

$$\frac{\cancel{\frac{1}{Z}} \exp\left\{-\sum_{c \in C_s} V_c(x_c)\right\} \cancel{\exp\left\{-\sum_{c \in \bar{C}_s} V_c(x_c)\right\}}}{\cancel{\frac{1}{Z}} \cancel{\exp\left\{-\sum_{c \in \bar{C}_s} V_c(x_c)\right\}} \sum_{x_s} \exp\left\{-\sum_{c \in C_s} V_c(x_c)\right\}}$$

$$= \frac{\exp\left\{-\sum_{c \in C_s} V_c(x_c)\right\}}{\sum_{x_s} \exp\left\{-\sum_{c \in C_s} V_c(x_c)\right\}}$$

$$= f(x_s, x_{\partial s})$$

$$= p(x_s \mid x_{\partial s})$$

# Lecture 16

$(\Rightarrow)$

1) Define
$$Q(x) = \log\left(\frac{p(x)}{p(0)}\right)$$

Then there exist functions $G$ so that

$$Q(x) = \sum_{1\le i\le n} x_i G_i(x_i)$$
$$+ \sum\sum_{1\le i<j\le n} x_i x_j G_{ij}(x_i, x_j)$$
$$+ \sum\sum\sum_{1\le i<j<k\le n} x_i x_j x_k G_{ijk}(x_i, x_j, x_k)$$
$$\vdots$$
$$+ x_1 x_2 \dots x_n G_{123\dots n}(x_1, x_2, \dots x_n)$$

) Define $\vec{X}_n = \{x_1, \dots x_{n-1}, 0, x_{n+1}, \dots x_n\}$

then
$$Q(X) - Q(\vec{X}_n) \equiv \log \frac{p(x_n, x_i\ i\ne n)}{p(x_n=0, x_i\ i\ne n)}$$

$$= \log \frac{p(x_n \mid X_{\partial n})\, p(x_i\ i\ne n)}{p(0 \mid X_{\partial n})\, p(x_i\ i\ne n)}$$

$$= \log\left(\frac{p(x_n \mid X_{\partial n})}{p(0 \mid X_{\partial n})}\right) = f(x_n, X_{\partial n})$$

2) If we subtract expressions for $Q(x)$ and $Q(\vec{X}_1)$, then terms not containing $\vec{X}_1$ must cancell

$$Q(X) - Q(\vec{X}_1) =$$

$$= X_1 \Big\{ G_1(X_1) + \sum_{2 \le j \le n} x_j \, G_{1j}(x_1, x_j) + \dots + x_2 x_3 \dots x_n \, G_{12\dots n}(x_1, \dots x_n) \Big\}$$

3. Choose $\ell \notin \partial 1$

Let $x_i = 0$ for $i \neq 1, \ell$

$$Q(x) - Q(x_1) = x_1 \{ G_1(x_1) + x_\ell G_{1\ell}(x_1, x_\ell) \}$$
$$= f(x_1, x_{\partial 1})$$

$\Rightarrow G_{1\ell} = 0$

4. Choose $\ell \in \partial 1$, $m \notin \partial 1$

Let $x_i = 0$ for $i \neq \ell, m, 1$

$$Q(x) - Q(x_1) = x_1 \{ G_1(x_1) + x_\ell G_{1\ell}(x_\ell) + x_\ell x_m G_{1\ell m}(x_1, x_\ell, x_m) \}$$

$$= f(x_1, x_{\partial 1})$$

So $\cancel{G_1} \cancel{x_1} x_\ell x_m G_{1\ell m}(x_1, x_\ell, x_m) = f(x_1, x_{\partial 1})$
$= f(x_1, x_\ell)$

$\Rightarrow \cancel{=} \cancel{x_1 G} x_\ell f'(x_1, x_\ell)$ form some $f'(x_1, x_\ell)$

but since when $x_m = 0 \Rightarrow x_1 x_\ell x_m G_{1\ell m} = 0$

$\Rightarrow x_1 x_\ell f'(x_1, x_\ell) = 0$

5. By induction,

if $k \notin \partial 1$ then

$$G_{1, i_2, \ldots, k, \ldots i_n} = 0$$

6. By a similar argument

if $i_k \notin \partial i_j$ then

$$G_{i_1 \ldots i_n} = 0$$

7)

$$V_c(x_c) \triangleq -G_c(x_c) \prod_{s \in c} x_s$$

$$Q(x) = \sum_{c \in \mathcal{C}} -V_c(x_c)$$

$$p(x) = p(0) \exp\left\{-\sum_{c \in \mathcal{C}} V_c(x_c)\right\}$$

$$= \frac{1}{Z} \exp\left\{\mp \sum_{c \in \mathcal{C}} V_c(x_c)\right\}$$

Q.E.D.

~~Example~~ GMRF

$$p(x) = \frac{1}{\sqrt{2\pi}^N} |B|^{-1/2} \exp\{-\tfrac{1}{2} X^T B X\}$$

$$-\frac{1}{2} X^T B X = \frac{1}{2} \sum_{i,j} x_i B_{ij} x_j$$

$$= \sum_{\{i,j\} \in C} x_i B_{ij} x_j + \sum_{i \in S} x_i^2 B_i$$

$$p(x) = \frac{1}{Z} \exp\Big\{ \sum_{\{i,j\} \in C} x_i B_{ij} x_j + \sum_{i \in S} B_i x_i^2 \Big\}$$

Only requires clique pairs
and singles

Lecture 17

Example

$\{X_i\}_{i=1}^{N}$ Markov Chain with strictly positive distribution.

$$p(X) = \prod_{i=2}^{N} p_i(x_i | x_{i-1})\, p_1(x_1)$$

Define $V_i(x_i, x_{i-1}) = -\log p(x_i | x_{i-1})$

$V_1(x_1) = -\log p(x_1)$

$$p(X) = \exp\left\{-\sum_{i=2}^{N} V_i(x_i, x_{i-1}) - V_1(x_1)\right\}$$

Gibbs distribution $\Rightarrow$ MRF

If $p_i(x_i | x_{i-1}) = \frac{\rho}{M-1} t(x_i, x_{i-1}) + (1-\rho)\,\delta(x_i, x_{i-1})$

$x_i \in [1, M]$

$$-\log p(x_i | x_{i-1}) = -\log\left\{\frac{\rho}{1-\rho}\frac{1}{M-1} t(x_i, x_{i-1}) + \delta(x_i, x_{i-1})\right\} - \log(1-\rho)$$

$$= \log\left((M-1)\frac{(1-\rho)}{\rho}\right) t(x_i, x_{i-1}) - \log(1-\rho)$$

$$= V_i(x_i, x_{i-1})$$

$$\beta \triangleq \log\left((M-1)\frac{(1-\rho)}{\rho}\right)$$

$$p(x) = \frac{1}{Z} \exp\left\{-\sum_{i=2}^{N} \beta\, t(x_i, x_{i-1}) + V_1(x_1)\right\}$$

$$p(x_n \mid x_i \; i \neq n) = \frac{p(x)}{\sum_{x_n} p(x_n, x_i \; i \neq n)}$$

$$= \frac{\frac{1}{Z} \exp\left\{ - \sum_{i=2}^{N} \beta t(x_i, x_{i-1}) + V_1(x_1) \right\}}{\sum_{x_n} \frac{1}{Z} \exp\left\{ - \sum_{i=2}^{N} \beta t(x_i, x_{i-1}) + V_1(x_1) \right\}}$$

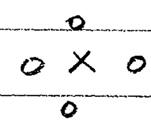
$$= \frac{\exp\left\{ -\beta \left( t(x_n, x_{n-1}) + t(x_{n+1}, x_n) \right) \right\}}{\sum_{k=0}^{M-1} \exp\left\{ -\beta \left( t(k, x_{n-1}) + t(x_{n+1}, k) \right) \right\}}$$

# The Ising Model

- Motivated by model for Magnetism

$$X_s = \begin{cases} 1 & \text{North pole} \\ -1 & \text{South pole} \end{cases}$$

- MRF with 4 pt neighborhood



- Let $r \in \partial s$ then

if $X_r X_s = 1 \Rightarrow$ poles are aligned $\Rightarrow$ low energy
$X_r X_s = -1 \Rightarrow$ pole are opposit $\Rightarrow$ high energy

Total energy $= U(X) = -\frac{J}{2} \sum_{\{r,s\} \in C} X_r X_s$

$C$ – set of all cliques
$J$ – physical constant

$$p(x) = P\{X = x\}$$

$$\text{Entropy} = S(p) = \sum_x -\log p(x)\, p(x)$$
$$= E[-\log p(X)]$$

$$\bar{\mathcal{E}} = \sum_x u(x)\, p(x) = E_p[u(X)]$$

If the system is in thermodynamic equilibrium then

$$p_e(x) = \underset{\substack{p \text{ s.t.} \\ E_p[u(x)] = \text{constant}}}{\operatorname{argmax}} S(p)$$

Maximize entropy while constraining energy $\Rightarrow$ Lagrange Multipliers

Solution

$$p(x) = \frac{1}{Z(T)} e^{-\frac{1}{KT} u(x)}$$

$K$ - Boltzmann's constant
$T$ - Temperature
$Z(T)$ - Partition function $= \sum_x e^{-\frac{1}{KT} u(x)}$

$$p(x) = \frac{1}{z} \exp\left\{ -\frac{1}{KT}\left( -\frac{J}{2} \sum_{\{r,s\} \in C} x_r x_s \right) \right\}$$

$$= \frac{1}{z} \exp\left\{ \frac{J}{2KT} \sum_{\{r,s\} \in C} x_r x_s \right\}$$

$$= \frac{1}{z} \exp\left\{ \frac{J}{KT} \sum_{\{r,s\} \in C} \left( \frac{1}{2} - t(x_r, x_s) \right) \right\}$$

$$t(k, l) = \begin{cases} 1 & k \neq l \\ 0 & \text{o.w} \end{cases}$$

$$= \frac{1}{z'} \exp\left\{ -\beta \sum_{\{r,s\} \in C} t(x_r, x_s) \right\}$$

$$\beta = \frac{J}{KT}$$

Note: Looks like the form of the 1-D Markov Chain

• What is the NonCausal dependence?

$$C_s = \{ c \in \mathcal{C} : x_s \in c \}$$
$$\bar{C}_s = \mathcal{C} - C_s$$

$$p(x_s | x_i \; i \neq s) = \frac{p(x_s, x_i \; i \neq s)}{\sum_{x_s} p(x_s, x_i \; i \neq s)}$$

$$= \frac{\frac{1}{Z} \exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\} \exp\left\{ -\beta \sum_{\{i,j\} \in \bar{C}_s} t(x_i, x_j) \right\}}{\frac{1}{Z} \exp\left\{ -\beta \sum_{\{i,j\} \in \bar{C}_s} t(x_i, x_j) \right\} \sum_{x_s} \exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\}}$$

$$= \frac{\exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\}}{\sum_{x_s} \exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\}}$$

Define

$$V(x_s, X_{\partial s}) = \sum_{\{i,j\} \in C_s} t(x_i, x_j)$$

= Number of neighbors in $X_{\partial s}$ not equal to $X_s$

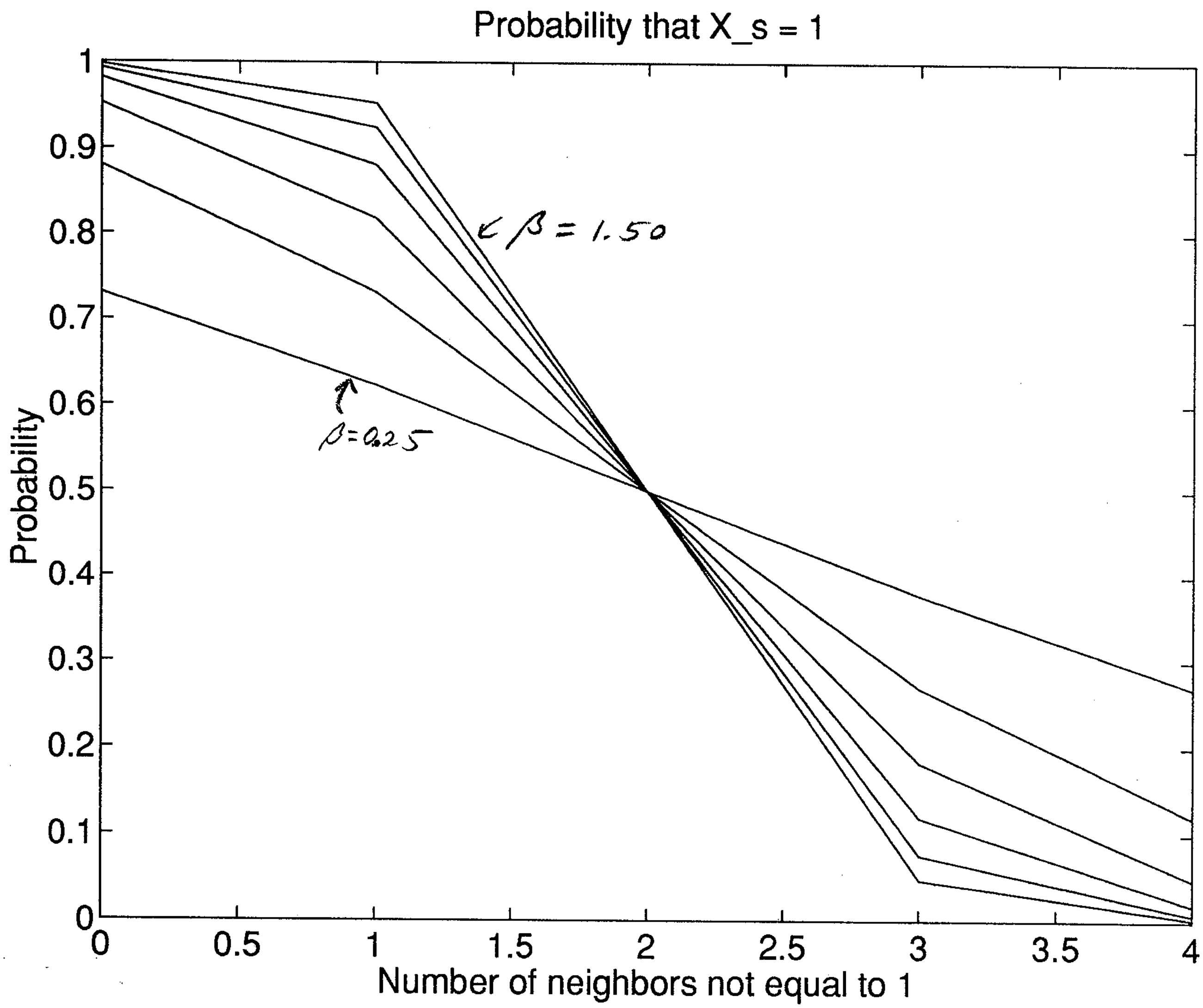
$p(x_s \mid x_i \; i \neq s) =$

$$\frac{\exp\{-\beta V(x_s, X_{\partial s})\}}{\exp\{-\beta V(x_s, X_{\partial s})\} + \exp\{-\beta V(-x_s, X_{\partial s})\}}$$

Notice that

$$V(-x_s, X_{\partial s}) = 4 - V(x_s, X_{\partial s})$$

$$p(x_s \mid x_i \; i \neq s) = \frac{1}{1 + \exp\{2\beta(V(x_s, X_{\partial s}) - 2)\}}$$


Probability that X_s = 1
Probability
Number of neighbors not equal to 1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5
4
$\beta = 1.50$
$\beta = 0.25$