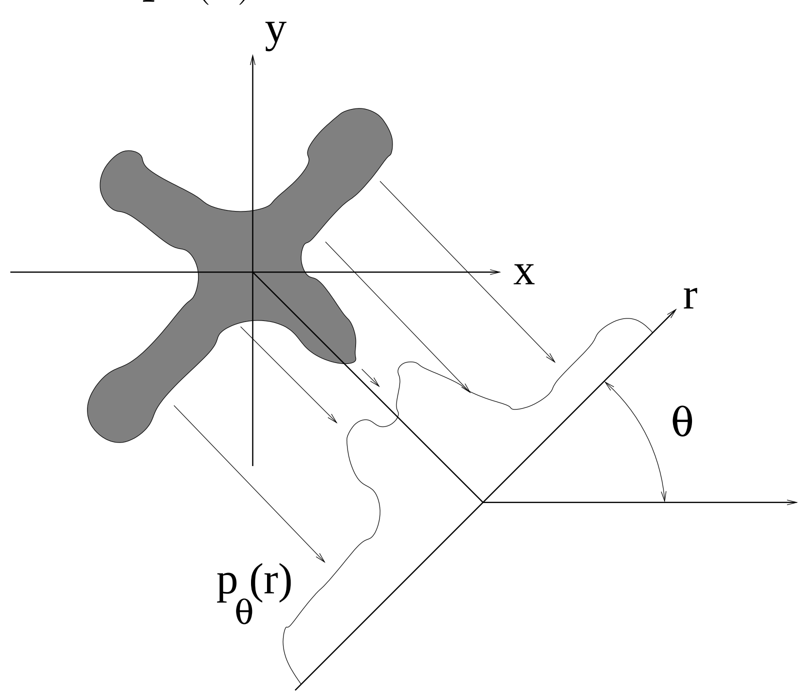
Application of Inverse Methods to Tomography

- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation

Forward Projection

- Typical tomographic imaging senerio:
  - Projections collected at every angle θ and displacement r.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.

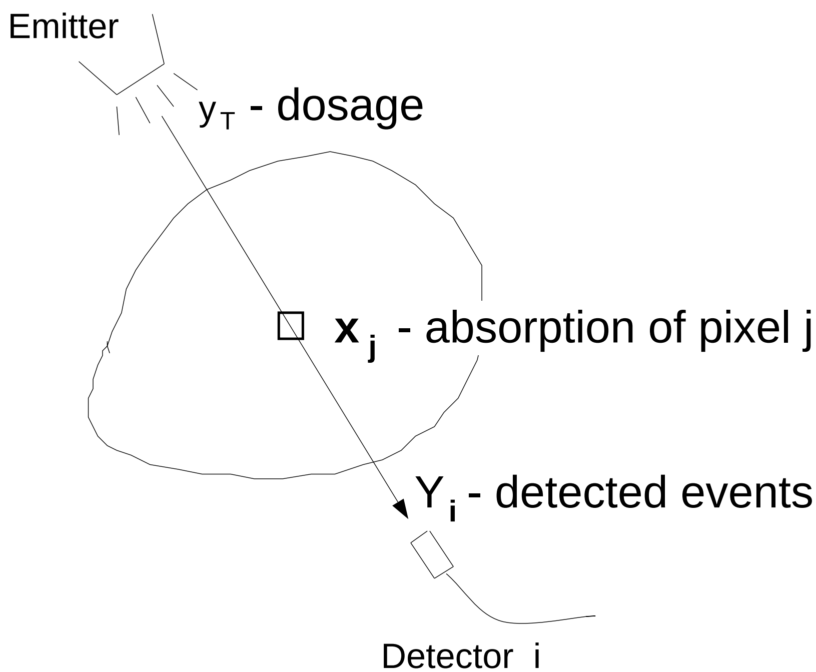


- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

Advantages of Iterative/Statistical Reconstruction

- Low signal-to-noise data
  - Data may vary with projection (dense objects, noisy detectors, etc.)
  - FBP treats all projections equally
- Missing projections
  - Dense objects may make some views impossible.
  - Helical scanners do not take every view at each position
- Complex geometries
  - Projections may be taken in fan-beam and cone-beam geometries
- Non-Gaussian prior modeling
  - Non-Gaussian models may be particularly appropriate for object cross-sections

Transmission Tomography



Y_T - Dosage emitted from source (not random)

X_j - j^{th} pixel

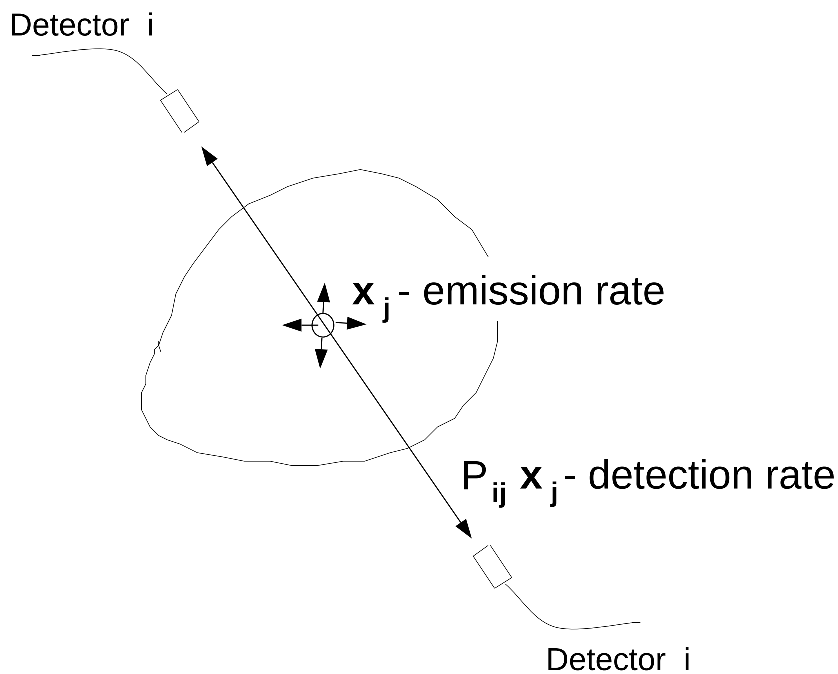
Y_i - Energy measured by i^{th} detector

P_{ij} - Contribution of j^{th} pixel to i^{th} detector

- Typical assumptions
  - Y_i are i.i.d. and Poisson
  - $E[Y_i|X] = Y_T \exp\{\Sigma_j P_{i,j} X_j\}$

- Includes computed tomography (CT), scanning electron microscope (SEM)

Emission Tomography



X_j - Emission rate from j^{th} pixel

Y_i - Energy measured by i^{th} detector *pair*

P_{ij} - Contribution of j^{th} pixel to i^{th} detector

- Typical assumptions
  - Y_i are i.i.d. and Poisson
  - $E[Y_i|X] = \Sigma_j P_{i,j} X_j$

- Includes positron emission tomography (PET), and single photon emission tomography (SPECT)

Statistical Data Model[3]

- Notation
  - y - vector of photon counts
  - x - vector of image pixels
  - P - projection matrix
  - $P_{j,*}$ - j^{th} row of projection matrix
- Emission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-P_{i*}x \; + y_i \log\{P_{i*}x\} - \log(y_i!)\right)$$

- Transmission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-y_T e^{-P_{i*}x} + y_i(\log y_T - P_{i*}x) - \log(y_i!)\right)$$

- Common form

$$\boxed{\log p(y|x) = -\sum_{i=1}^{M} f_i(P_{i*}x)}$$

  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!

Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

$$
\begin{aligned}
\hat{x} &= \arg\max_{x} p(x|y) \\
&= \arg\max_{x>0} \left\{ -\sum_{i=1}^{M} f_i(P_{i*}x) - \sum_{k<j} b_{k,j}\, \rho(x_k - x_j) \right\}
\end{aligned}
$$

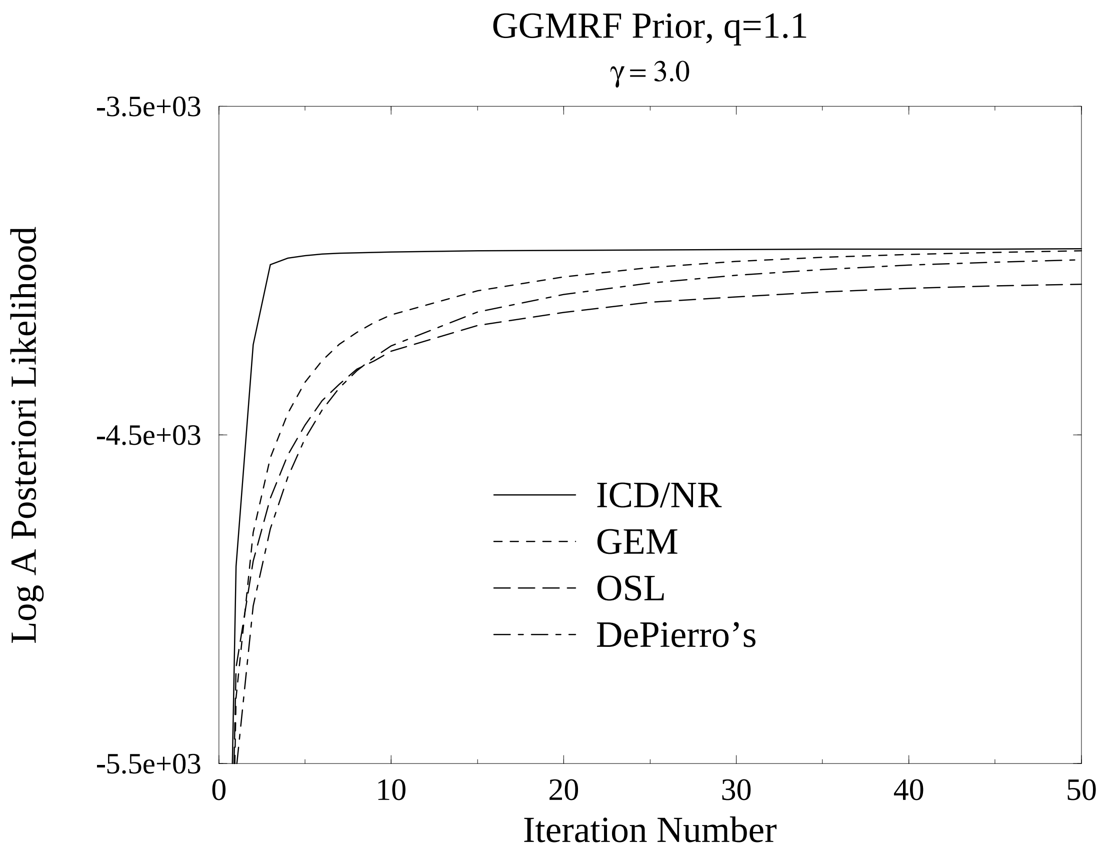
- Can be solved using direct optimization
- Incorporates positivity constraint

MAP Optimization Strategies

- Expectation maximization (EM) based optimization strategies
  - ML reconstruction[12, 10]
  - MAP reconstruction[8, 7, 9]
  - Slow convergence; Similar to gradient search.
  - Accelerated EM approach[6]
- Direct optimization
  - Preconditioned gradient descent with soft positivity constraint[5]
  - ICM iterations (also known as ICD and Gauss-Seidel)[3]

Convergence of ICM Iterations: MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel



- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green's (OSL), and Hebert/Leahy's GEM, and De Pierro's method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$.

Estimation of σ from Tomographic Data

- Assume a GGMRF prior distribution of the form

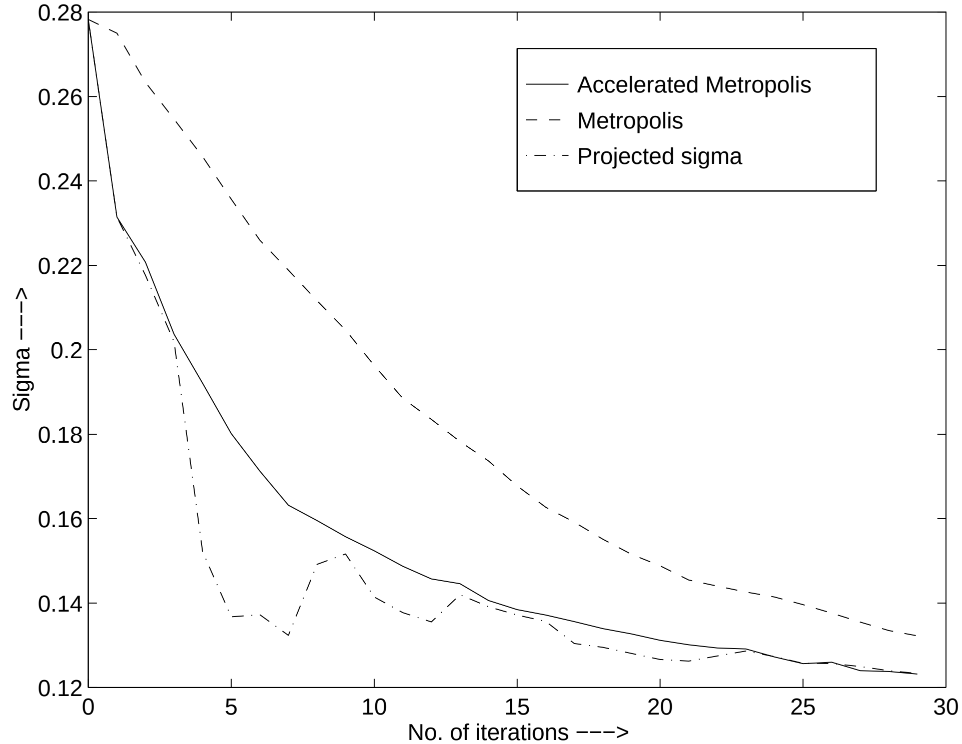
$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ \frac{1}{p\sigma^p} U(x) \right\}$$

- Problem: We don't know X!
- EM formulation for incomplete data problem

$$\begin{aligned} \sigma^{(k+1)} &= \arg\max_{\sigma} E\left\{ \log p(X|\sigma) | Y = y, \sigma^{(k)} \right\} \\ &= \left(E\left\{ \frac{1}{N} U(X) | Y = y, \sigma^{(k)} \right\} \right)^{1/p} \end{aligned}$$

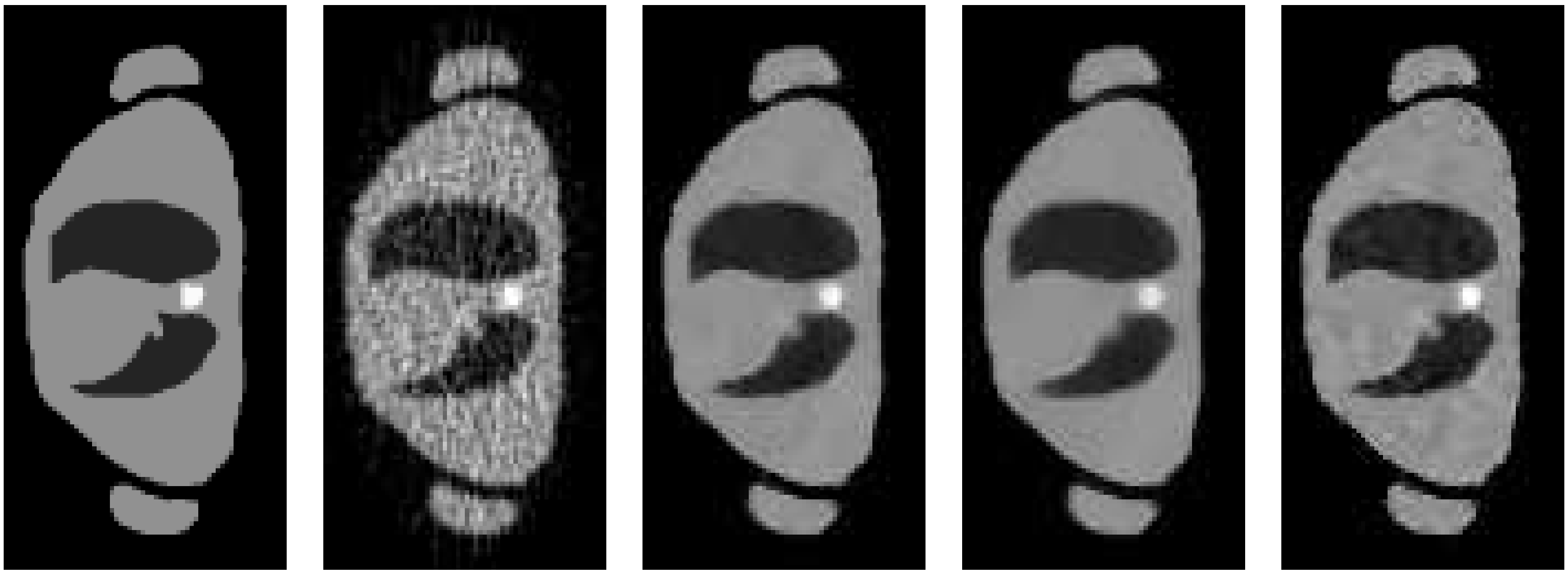
- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.

Example of Estimation of σ from Tomographic Data



- The above plot shows the EM updates for σ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter s denotes the standard deviation of the symmetric transition distribution for the CM method.

Example of Tomographic Reconstructions

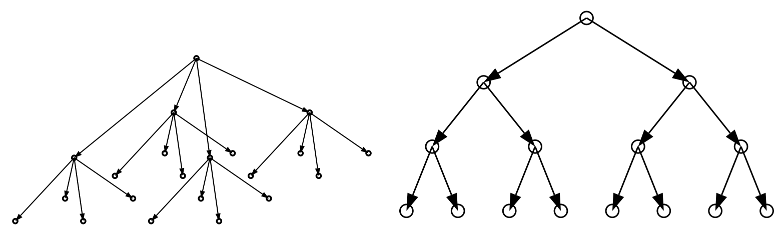


a	b	c	d	e

- (a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter σ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$
- Phantom courtesy of J. Fessler, University of Michigan

Multiscale Stochastic Models

- Generate a Markov chain in scale



- Some references
  - Continuous models[2, 1, 11]
  - Discrete models[4, 11]
- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM's.

References

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