Application of Inverse Methods to Tomography

- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation
Forward Projection

- Typical tomographic imaging scenario:
  - Projections collected at every angle $\theta$ and displacement $r$.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.

- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)
Advantages of Iterative/Statistical Reconstruction

- Low signal-to-noise data
  - Data may vary with projection (dense objects, noisy detectors, etc.)
  - FBP treats all projections equally
- Missing projections
  - Dense objects may make some views impossible.
  - Helical scanners do not take every view at each position
- Complex geometries
  - Projections may be taken in fan-beam and cone-beam geometries
- Non-Gaussian prior modeling
  - Non-Gaussian models may be particularly appropriate for object cross-sections
Transmission Tomography

- $Y_T$ - Dosage emitted from source (not random)
- $X_j$ - $j^{th}$ pixel
- $Y_i$ - Energy measured by $i^{th}$ detector
- $P_{ij}$ - Contribution of $j^{th}$ pixel to $i^{th}$ detector

• Typical assumptions
  - $Y_i$ are i.i.d. and Poisson
  - $E[Y_i|X] = Y_T \exp \{\sum_j P_{i,j}X_j\}$

• Includes computed tomography (CT), scanning electron microscope (SEM)
Emission Tomography

\[ X_j \] - Emission rate from \( j^{th} \) pixel

\[ Y_i \] - Energy measured by \( i^{th} \) detector

\[ P_{ij} \] - Contribution of \( j^{th} \) pixel to \( i^{th} \) detector

- Typical assumptions
  
  - \( Y_i \) are i.i.d. and Poisson
  
  - \( E[Y_i|X] = \sum_j P_{i,j}X_j \)

- Includes positron emission tomography (PET), and single photon emission tomography (SPECT)
Statistical Data Model[3]

• Notation
  - $y$ - vector of photon counts
  - $x$ - vector of image pixels
  - $P$ - projection matrix
  - $P_{j,*}$ - $j^{th}$ row of projection matrix

• Emission formulation
  \[
  \log p(y|x) = \sum_{i=1}^{M} (-P_{i,*}x + y_i \log \{P_{i,*}x\} - \log(y_i!))
  \]

• Transmission formulation
  \[
  \log p(y|x) = \sum_{i=1}^{M} (-y_T e^{-P_{i,*}x} + y_i (\log y_T - P_{i,*}x) - \log(y_i!))
  \]

• Common form
  \[
  \log p(y|x) = - \sum_{i=1}^{M} f_i(P_{i,*}x)
  \]
  
  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!
Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

\[ \hat{x} = \arg \max_x p(x|y) \]

\[ = \arg \max_{x>0} \left\{- \sum_{i=1}^{M} f_i(P_i x) - \sum_{k<j} b_{k,j} \rho(x_k - x_j)\right\} \]

- Can be solved using direct optimization
- Incorporates positivity constraint
MAP Optimization Strategies

• Expectation maximization (EM) based optimization strategies
  – ML reconstruction[12, 10]
  – MAP reconstruction[8, 7, 9]
  – Slow convergence; Similar to gradient search.
  – Accelerated EM approach[6]

• Direct optimization
  – Preconditioned gradient descent with soft positivity constraint[5]
  – ICM iterations (also known as ICD and Gauss-Seidel)[3]
Convergence of ICM Iterations:
MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel

![Graph showing convergence of ICM iterations with various methods and priors.](image)

- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green’s (OSL), and Hebert/Leahy’s GEM, and De Pierro’s method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$. 
Estimation of $\sigma$ from Tomographic Data

- Assume a GGMRF prior distribution of the form
\[
p(x) = \frac{1}{\sigma^N Z(1)} \exp \left\{ \frac{1}{p\sigma^p} U(x) \right\}
\]

- Problem: We don’t know $X$!

- EM formulation for incomplete data problem
\[
\sigma^{(k+1)} = \arg \max_{\sigma} E \left\{ \log p(X|\sigma) | Y = y, \sigma^{(k)} \right\}
= \left( E \left\{ \frac{1}{N} U(X) | Y = y, \sigma^{(k)} \right\} \right)^{1/p}
\]

- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.
Example of Estimation of $\sigma$ from Tomographic Data

- The above plot shows the EM updates for $\sigma$ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter $s$ denotes the standard deviation of the symmetric transition distribution for the CM method.
Example of Tomographic Reconstructions

(a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter $\sigma$ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$

- Phantom courtesy of J. Fessler, University of Michigan
Multiscale Stochastic Models

- Generate a Markov chain in scale

- Some references
  - Continuous models[2, 1, 11]
  - Discrete models[4, 11]

- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM’s.
References


