Continuous State MRF's

- Topics to be covered:
  - Quadratic functions
  - Non-Convex functions
  - Continuous MAP estimation
  - Convex functions

Why use Non-Gaussian MRF's?

- Gaussian MRF's do not model edges well.
- In applications such as image restoration and tomography, Gaussian MRF's either
  - Blur edges
  - Leave excessive amounts of noise

Gaussian MRF's

- Zero mean Gaussian MRF's have density functions with the form

$$p(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{2\sigma^2} x^t B x \right\}$$

- It can be shown that

$$x^t B x = \sum_{s \in S} a_s x_s^2 + \sum_{\{s,r\} \in C} b_{sr} |x_s - x_r|^2$$

where

$$a_s \triangleq \sum_{r \in S} B_{s,r}$$

$$b_{s,r} \triangleq -B_{s,r}$$

- We will further assume that $a_s = 0$ and $\sum_r b_{sr} = 1$, so that

$$\log p(x) = -\frac{1}{2\sigma^2} \sum_{\{s,r\} \in C} b_{sr} |x_s - x_r|^2 - \log Z$$

MAP Estimation with Gaussian MRF's

- MAP estimate has the form

$$\hat{x} = \arg\min_{x} \left\{ -\log p(y|x) + \sum_{\{s,r\} \in C} b_{sr} |x_s - x_r|^2 \right\}$$

- **Problem:**
  - The terms $|x_s - x_r|^2$ penalize rapid changes in gray level.
  - Quadratic function, $|\cdot|^2$, excessively penalizes image edges.

Non-Gaussian MRF's Based on Pair-Wise Cliques

- We will consider MRF's with pair-wise cliques

$$p(x) = \frac{1}{Z} \exp\left\{ - \sum_{\{s,r\} \in C} b_{sr} \rho\left(\frac{x_s - x_r}{\sigma} \right) \right\}$$

$|x_s - x_r|$ - is the change in gray level.

σ - controls the gray level variation or scale.

$\rho(\Delta)$:

- Known as the potential function.
- Determines the cost of abrupt changes in gray level.
- $\rho(\Delta) = |\Delta|^2$ is the Gaussian model.

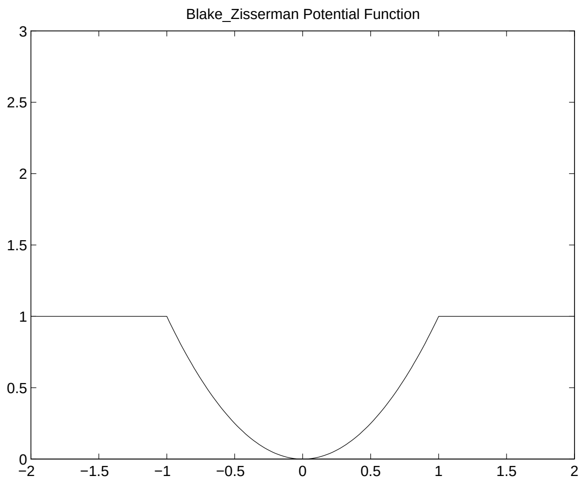
$\rho'(\Delta) = \frac{d\rho(\Delta)}{d\Delta}$:

- Known as the influence function from "M-estimation"[14, 11].
- Determines the attraction of a pixel to neighboring gray levels.

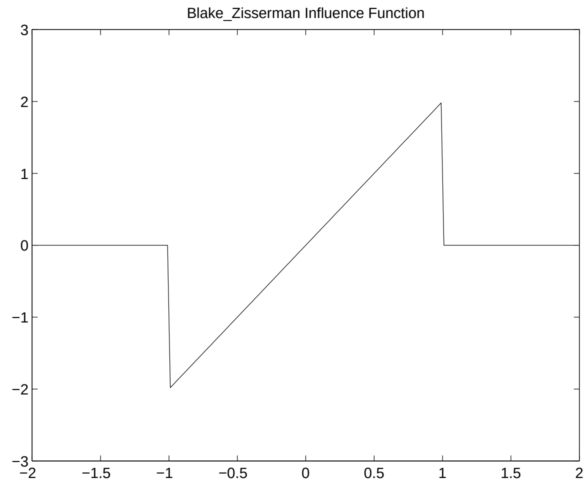
Weak Spring Model

- Proposed by Blake and Zisserman [3, 2] as a model of a "weak spring" that can break if the values of adjacent pixels differ too much.

$$\rho(\Delta) = \min\left\{\Delta^2, 1\right\}$$



Potential Function



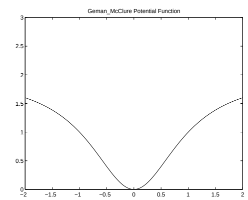
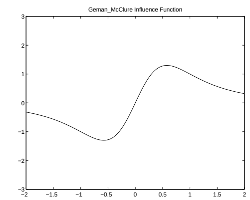
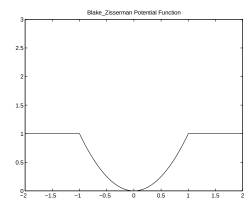
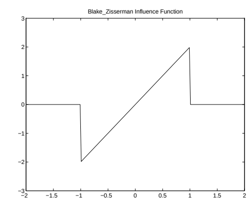
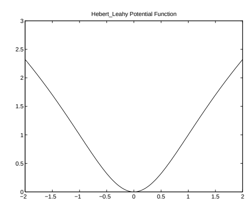
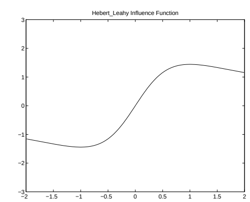
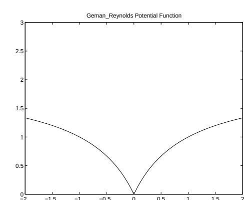
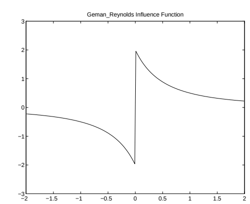
Influence Function

- T - Edge magnitude

$$\Delta > T \Rightarrow \text{no attraction from influence function}$$

$$\Delta < T \Rightarrow \text{Gaussian smoothing}$$

Non-Convex Potential Functions

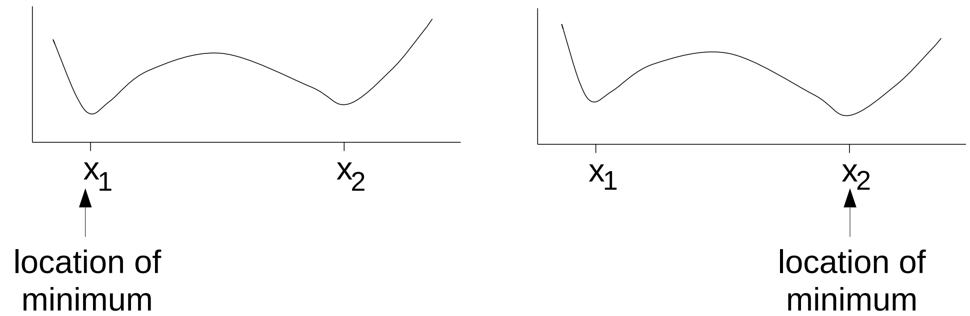
Authors	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Geman and McClure	$\frac{\Delta^2}{1+\Delta^2}$	[7, 8]		
Blake and Zisserman	$\min\left\{\Delta^2, 1\right\}$	[3, 2]		
Hebert and Leahy	$\log\left(1+\Delta^2\right)$	[10]		
Geman and Reynolds	$\frac{\|\Delta\|}{1+\|\Delta\|}$	[6]		

Properties of Non-Convex Potential Functions

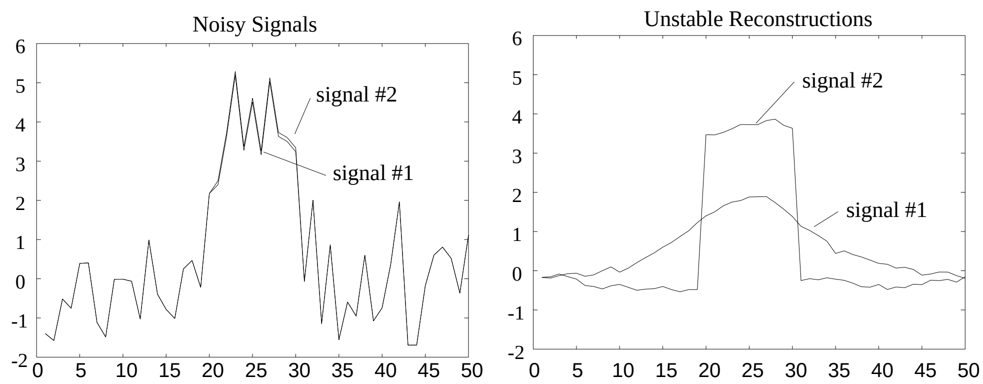
- Advantages
  - Very sharp edges
  - Very general class of potential functions
- Disadvantages
  - Difficult (impossible) to compute MAP estimate
  - Usually requires the choice of an edge threshold
  - **MAP estimate is a discontinuous function of the data**

Continuous (Stable) MAP Estimation[4]

- Minimum of non-convex function can change abruptly.

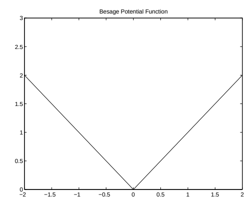
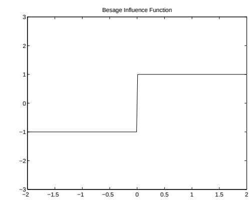
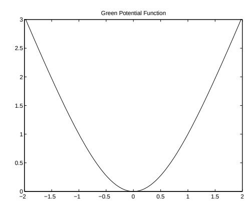
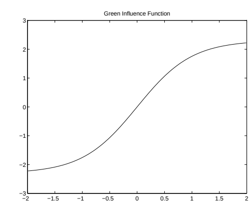
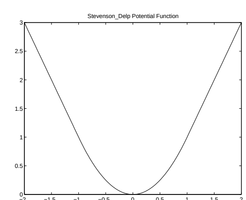
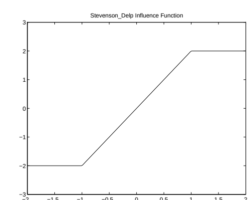
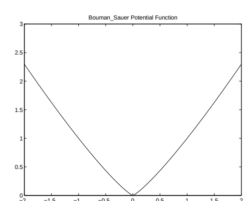
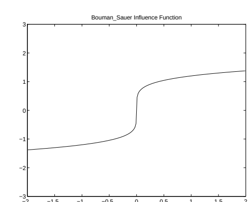


- Discontinuous MAP estimate for Blake and Zisserman potential.



- Theorem:[4] - If the log of the posterior density is **strictly convex**, then the MAP estimate is a continuous function of the data.

Convex Potential Functions

Authors(Name)	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Besag	$\lvert\Delta\rvert$	[1]		
Green	$\log\cosh\Delta$	[9]		
Stevenson and Delp (Huber function)	$\min\{\lvert\Delta\rvert^2, 2\lvert\Delta\rvert - 1\}$	[17]		
Bouman and Sauer (Generalized Gaussian MRF)	$\lvert\Delta\rvert^p$	[4]		

Properties of Convex Potential Functions

- Both $\log\cosh(\Delta)$ and Huber functions
  - Quadratic for $|\Delta| << 1$
  - Linear for $|\Delta| >> 1$
  - Transition from quadratic to linear determines edge threshold.
- Generalized Gaussian MRF (GGMRF) functions
  - Include $|\Delta|$ function
  - Do not require an edge threshold parameter.
  - Convex and differentable for $p > 1$.

Parameter Estimation for Continuous MRF's

- Topics to be covered:
  - Estimation of scale parameter, σ
  - Estimation of temperature, T, and shape, p

ML Estimation of Scale Parameter, σ, for Continuous MRF's [5]

- For any continuous state Gibbs distribution

$$p(x) = \frac{1}{Z(\sigma)} \exp\left\{-U(x/\sigma)\right\}$$

the partition function has the form

$$Z(\sigma) = \sigma^N Z(1)$$

- Using this result the ML estimate of σ is given by

$$\frac{\sigma}{N} \frac{d}{d\sigma} U(x/\sigma) \Big|_{\sigma=\hat{\sigma}} - 1 = 0$$

- This equation can be solved numerically using any root finding method.

ML Estimation of σ for GGMRF's [12, 5]

- For a Generalized Gaussian MRF (GGMRF)

$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ -\frac{1}{p\sigma^p} U(x) \right\}$$

where the energy function has the property that for all $\alpha > 0$

$$U(\alpha x) = \alpha^p U(x)$$

- Then the ML estimate of σ is

$$\hat{\sigma} = \left(\frac{1}{N} U(x) \right)^{(1/p)}$$

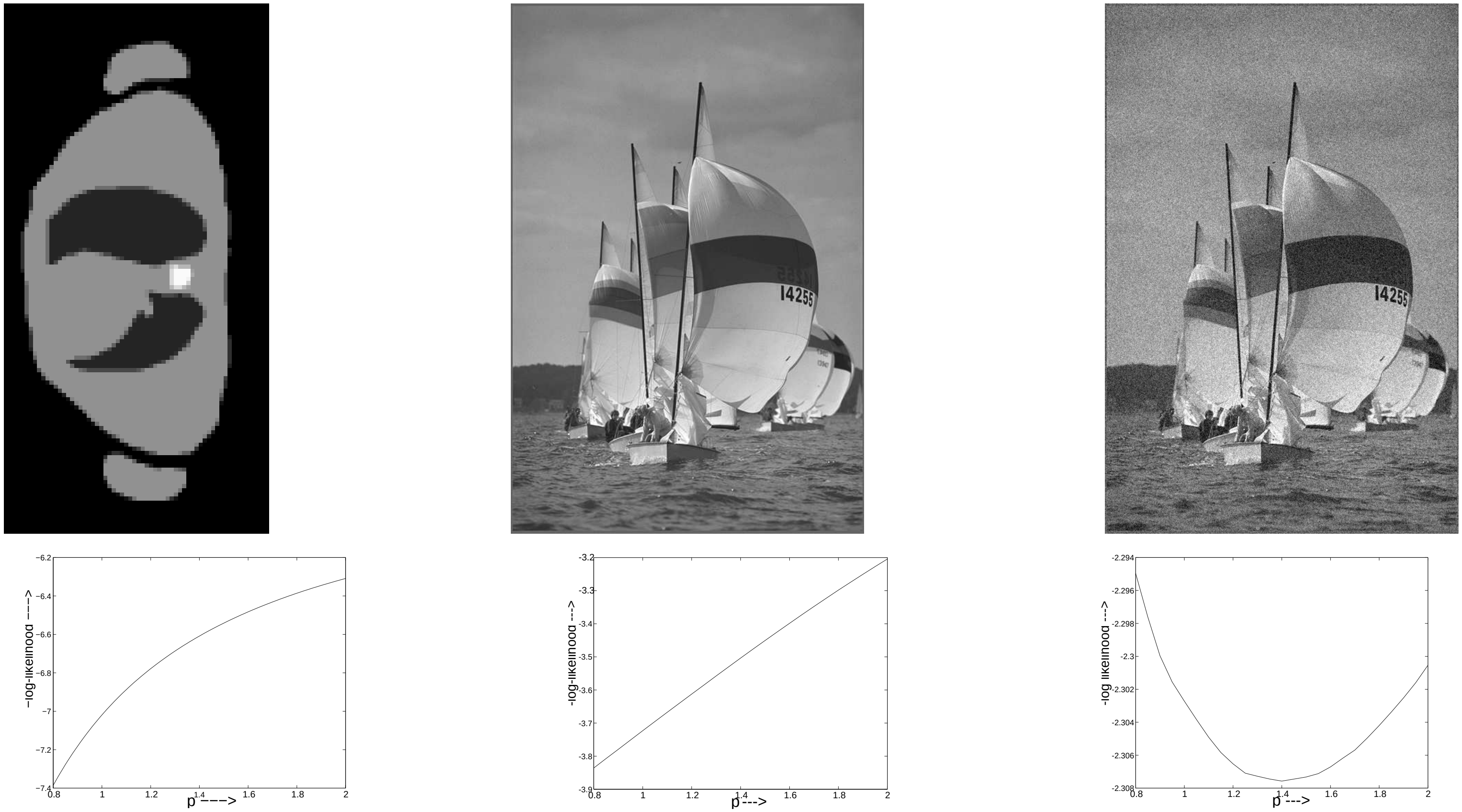
- Notice for that for the i.i.d. Gaussian case, this is

$$\hat{\sigma} = \sqrt{\frac{1}{N} \sum_s |x_s|^2}$$

Estimation of Temperature, T, and Shape, p, Parameters

- ML estimation of T[8]
  - Used to estimate T for any distribution.
  - Based on "off line" computation of log partition function.
- Adaptive method [13]
  - Used to estimate p parameter of GGMRF.
  - Based on measurement of kurtosis.
- ML estimation of p[16, 15]
  - Used to estimate p parameter of GGMRF.
  - Based on "off line" computation of log partition function.

Example Estimation of p Parameter



(a) (b) (c)

- ML estimation of p for (a) transmission phantom (b) natural image (c) image corrupted with Gaussian noise. The plot below each image shows the corresponding negative log-likelihood as a function of p. The ML estimate is the value of p that minimizes the plotted function.

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