PURDUE

ECE 63700

Final Exam, May 7, Spring 2025

NAME ______________________________________

PUID ____

Exam instructions:

- A fact sheet is included at the end of this exam for your use.
- You have 120 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

To ensure Gradescope can read your exam:

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

Name/PUID: Key

Problem 1. (35pt) Linear Time Invariant Systems

Let $T : \Re^N \to \Re^M$ be a function.

Linear function: We say that T is linear if $\forall \alpha \in \Re$, $\forall \beta \in \Re$, $\forall x \in \Re^N$, and $\forall y \in \Re^N$,

$$T[\alpha x + \beta y] = \alpha T[x] + \beta T[y] \ .$$

Infinite dimensional functions: When $N = M = \infty$, then the arguments of T are discrete time functions. In other words, $x \in \mathbb{R}^\infty$, so then $x_n \in \mathbb{R}$ where $n \in \{\ldots, -1, 0, 1, \ldots\}$.[1]

Shifting function: Define the shifting function $S_k[x] = z$ where $x, z \in \mathbb{R}^\infty$ and $z_n = x_{n-k}$. So in other words, S_k delays its input by k samples.

Time-invariant functions: Then we say that T is time invariant if for all $x, z \in \mathbb{R}^\infty$ and for all k, we have that

$$S_k[T[x]] = T[S_k[x]] \ .$$

Problem 1a) Assume $N, M < \infty$ and T is linear, then what is the most general form of the function $T[x]$?

Solution:

$$T[x] = Ax \ ,$$

for some matrix A is the most general form of a finite dimensional linear function.

Problem 1b) Assume $N, M < \infty$ and T is linear, then is $T[x]$ time invariant?

Solution: No, T cannot be time invariant because $S[x]$ requires that $x \in \mathbb{R}^\infty$. Intuitively, you can only shift a function x_n if it goes to infinity.

Problem 1c) Assume $N, M < \infty$, then give an example of a **nonlinear** function T.

Solution: Let $y = T[x]$, then let $y_n = x_n^2$ is a nonlinear function.

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[1] For this case, we also assume that T is closed, that is to say that for any sequence of inputs, $x_n \to x$, then $\lim_{n\to\infty} T(x_n) = T(x)$.

Problem 1d) Assume $N, M = \infty$ and T is linear, then what is the most general form of the function $T[x]$?

Solution: Let $y = T[x]$, then the most general form is

$$y_n = \sum_{m=-\infty}^{\infty} h_{n,m} x_m \ ,$$

for some function $h_{n,m}$.

Problem 1e) Assume $N, M = \infty$ and T is linear and time-invariant, then what is the most general form of the function $T[x]$?

Solution: Let $y = T[x]$, then the most general form is

$$y_n = \sum_{m=-\infty}^{\infty} h_{n-m} x_m \ ,$$

for some function h_n.

Problem 1f) Assume $N, M = \infty$ and T time-invariant, and that the input is given by $x_n = e^{j\omega n}$, then what is the most general form of the output?

Solution: Let $y = T[x]$ where $x_n = e^{j\omega n}$, then the most general form is

$$y_n = C e^{j\omega n} \ ,$$

where C is a complex number, and in general $C = H(e^{j\omega})$ the DTFT of the impulse response h_n.

Problem 1g)

Assume $N, M = \infty$ and T time-invariant, then give an example of a **nonlinear** function T.

Solution: Let $y = T[x]$, then let $y_n = x_n^2$ is a nonlinear function that is time-invariant.

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Problem 2. (40pt) Image Sharpening

Consider an LSI system H with an impulse response of

$$h(m,n) = \frac{1}{4} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \left(\frac{1}{2}\right)^{|k|+|l|} \delta(m-k, n-l) \ ,$$

and a second LSI system G with an impulse response of

$$g(m,n) = \delta(m,n) + \lambda \left[\delta(m,n) - h(m,n)\right] \ ,$$

for some scalar parameter $\lambda \in \mathbb{R}$.

Problem 2a) Is $h(m,n)$ a separable function? Justify your answer.

Solution: Yes, $h(m,n) = h_1(m)h_1(n)$ is a separable function with

$$h_1(n) = \frac{1}{2} \sum_{k=-1}^{1} \left(\frac{1}{2}\right)^{|k|} \delta(n-k) \ .$$

Problem 2b) What is the DC gain of the filter H? Justify your answer.

Solution: The DC gain is given by $H(e^{j\mu}, e^{j\nu}) = \sum_{m,n} h(m,n) = 1$.

Problem 2c) Calculate, $H(e^{j\mu}, e^{j\nu})$, the CSFT of $h(m,n)$.

Solution: Let $h(m,n) = h_1(m)h_1(n)$ where

$$h_1(n) = \frac{1}{2} \sum_{k=-1}^{1} \left(\frac{1}{2}\right)^{|k|} \delta(n-k) \ .$$

Then the CTFT of h_1 is given by

$$\begin{aligned} H_1(e^{j\omega}) &= \frac{1}{2} \left\{1 + (1/2)(e^{j\omega} + e^{-j\omega})\right\} \\ &= \frac{1}{2}(1 + \cos(\omega)) \ . \end{aligned}$$

So we have that

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4} (1 + \cos(\mu)) (1 + \cos(\nu)) \ .$$

Problem 2d) Calculate, $G(e^{j\mu}, e^{j\nu})$, the CSFT of $g(m,n)$.

Solution:

$$G(e^{j\mu}, e^{j\nu}) = 1 + \lambda \left[1 - \frac{1}{4} (1 + \cos(\mu)) (1 + \cos(\nu)) \right] \ .$$

Problem 2e) What is the DC gain of the filter G? Justify your answer.

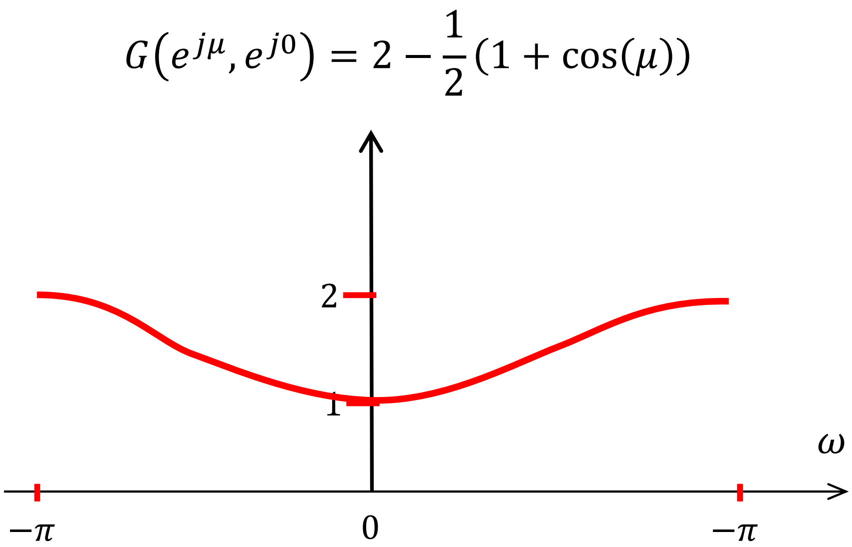
Solution: The DC gain is given by

$$\begin{aligned} G(e^{j0}, e^{j0}) &= 1 + \lambda \left[1 - \frac{1}{4} (1 + \cos(0)) (1 + \cos(0)) \right] \\ &= 1 + \lambda \left[1 - \frac{1}{4} (1 + 1) (1 + 1) \right] \\ &= 1 + \lambda [1 - 1] \\ &= 1 \ . \end{aligned}$$

Problem 2f) Sketch the function $G(e^{j\mu}, e^{j\nu})$, for $\nu = 0$ and $\lambda = 1.0$.

Solution:

$$\begin{aligned} G(e^{j\mu}, e^{j0}) &= 1 + \lambda \left[1 - \frac{1}{4} (1 + \cos(\mu)) (1 + \cos(0)) \right] \\ &= 1 + \lambda \left[1 - \frac{1}{2} (1 + \cos(\mu)) \right] \\ &= 1 + 1.0 \left[1 - \frac{1}{2} (1 + \cos(\mu)) \right] \\ &= 2 - \frac{1}{2} (1 + \cos(\mu)) \ . \end{aligned}$$



Problem 2g) What does G do for $\lambda > 0$?

Solution: For $\lambda > 0$, G is a sharpening filter.

Problem 2h) What does G do for $\lambda < 0$?

Solution: For $\lambda < 0$, G is a blurring filter.

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Problem 3. (40pt) Up Sampling

Let $s(t)$ be a continuous time function that is band limited to a maximum frequency of f_c. Then let $x(n) = s(nT)$ be a sampled version of s where $T = 1/f_s$ is the sampling period, and f_s is the sampling frequency in Hz. Define $T_{nyquest}$ as the maximum value of T that ensures that $x(n)$ contains all the information in $s(t)$.

Also, let $y(n) = s(nT/L)$ be a version that is sampled at L times the sampling rate for an integer $L \geq 2$.

Problem 3a) What is the value of $T_{nyquest}$?

Solution:

$$T_{nyquest} = \frac{1}{2f_c} \ .$$

Problem 3b) If $T < T_{nyquest}$, then can $f(t)$ also be reconstructed from $y(n)$? Justify your answer.

Solution: Yes, because we know that $x(n) = y(Ln)$, and we know that we can reconstruction $f(t)$ from $x(n)$.

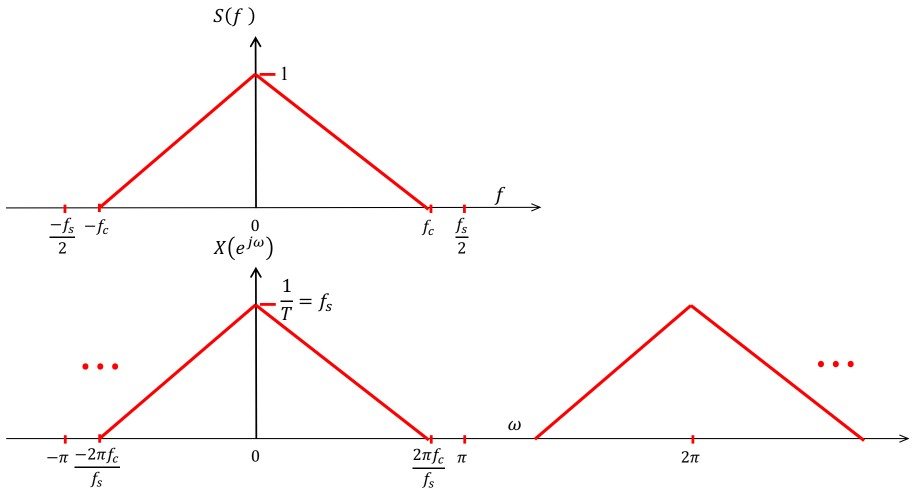
Problem 3c) Calculate an expression for $X(e^{j\omega})$ in terms of $S(f)$.

Solution:

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} S\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

Problem 3d) Sketch the spectra of $S(f)$ and $X(e^{j\omega})$.

Solution:



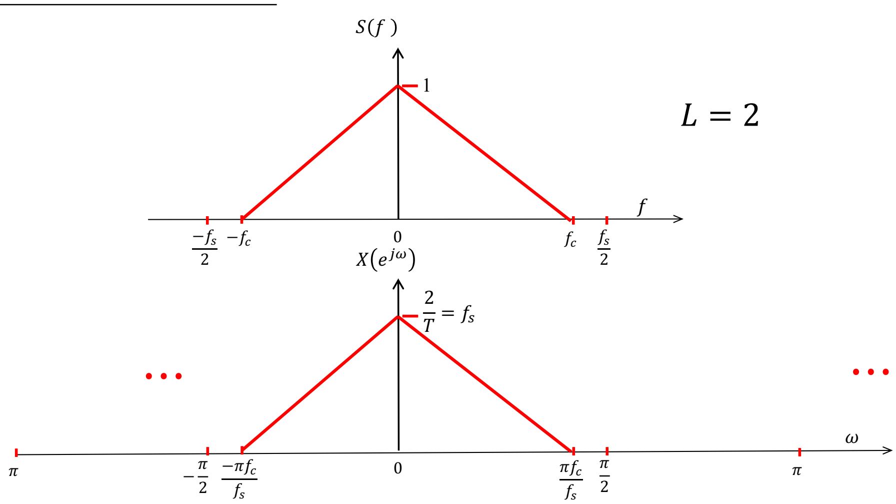
Problem 3e) Calculate an expression for $Y(e^{j\omega})$ in terms of $S(f)$.

Solution:

$$Y(e^{j\omega}) = \frac{L}{T} \sum_{k=-\infty}^{\infty} S\left(\frac{\omega - 2\pi k}{2\pi T/L}\right)$$

Problem 3f) Sketch the spectra of $S(f)$ and $Y(e^{j\omega})$ for $L = 2$.

Solution:



Problem 3g) Calculate an expression for $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$.

Solution:

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega L})$$

where $H(e^{j\omega}) = \text{rect}(\omega L)$ for $|\omega| < \pi$.

Problem 3h) Describe the steps required to compute $y(n)$ from $x(n)$. Provide a detailed specification of each step.

Solution: First you should up sample $x(n)$ to from $z(n)$.

$$z(n) = \begin{cases} x(n/L) & \text{for } n \mod L = 0 \\ 0 & \text{for } n \mod L \neq 0 \end{cases} .$$

Then filter the signal with the filter $h(n) = L\text{sinc}(n/L)$ to yield

$$y(n) = x(n) * h(n) \ .$$

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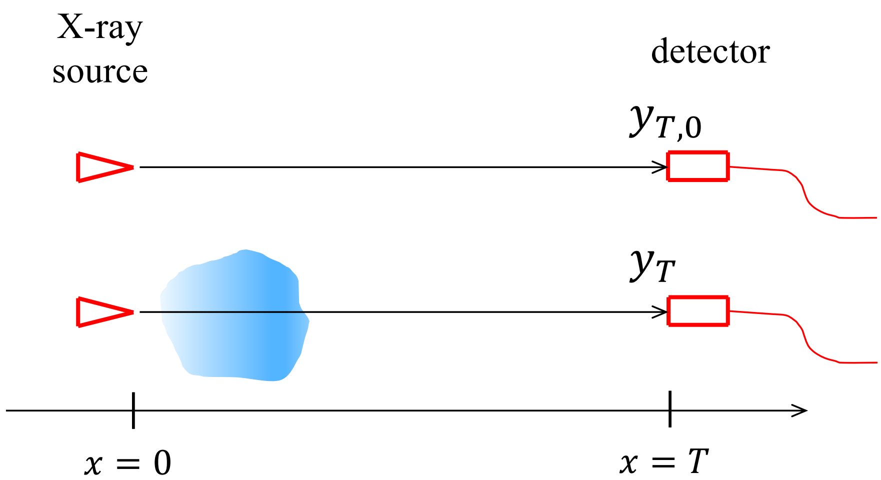
Problem 4. (35pt) Computed Tomography

Consider the X-ray transmission imaging system pictured below with a mono-energetic source.



We know that the photon flux, $y(x)$, as a function of depth, x, (units of cm) obeys the following equation

$$\frac{dy(x)}{dx} = -\mu(x)y(x) \ ,$$

where $\mu(x)$ (units of cm^{-1}) is the X-ray density of the material at location x along the path.

Define, $y_T = y(T)$ and $y_0 = y(0)$ where $y(x)$ is the photon flux with the object present.

Furthermore, assume that you are imaging a solid objected made from polyethylene. Then let $I(x)$ be an indicator function so that $I(x) = 1$ indicates that the polyethylene is present at location x, and $I(x) = 0$ indicates that it is absent. In this case, we have that

$$\mu(x) = \mu_E I(x) \ ,$$

where

$$\mu_E = \begin{cases} 0.21\text{cm}^{-1} & \text{at 40 keV} \\ 0.17\text{cm}^{-1} & \text{at 80 keV} \end{cases} .$$

Also define

$$L = \int_0^T I(x)dx \ ,$$

where L is the path length in cm through the polyethylene material.

Problem 4a) Find a solution to the differential equation, $y(x)$, which meets the boundary condition that $y(0) = y_0$.

Solution:

$$y(x) = y_0 \exp\left\{ -\int_0^x \mu(t)dt \right\}$$

Problem 4b) Show that if the object is removed (i.e., $\mu(x) = 0$), then the measurement is given by $y_{T,0} = y_0$.

Solution:

$$\begin{aligned} y(x) &= y_0 \exp\left\{ -\int_0^x \mu(t)dt \right\} \\ &= y_0 \exp\left\{ -\int_0^x 0\, dt \right\} \\ &= y_0 \exp\{-0\} \\ &= y_0 \ . \end{aligned}$$

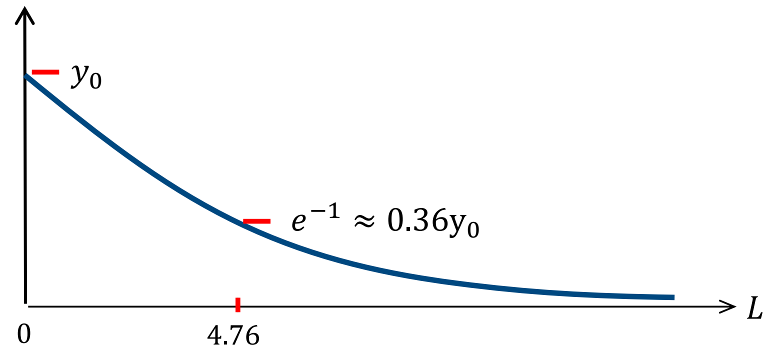
Problem 4c) Write an expression for the measurement y_T in terms of μ_E and L.

Solution:

$$\begin{aligned} y_T &= y(T) \\ &= y_0 \exp\left\{ -\int_0^T \mu_E I(t)dt \right\} \\ &= y_0 \exp\left\{ -\mu_E \int_0^T I(t)dt \right\} \\ &= y_0 \exp\{-\mu_E L\} \ . \end{aligned}$$

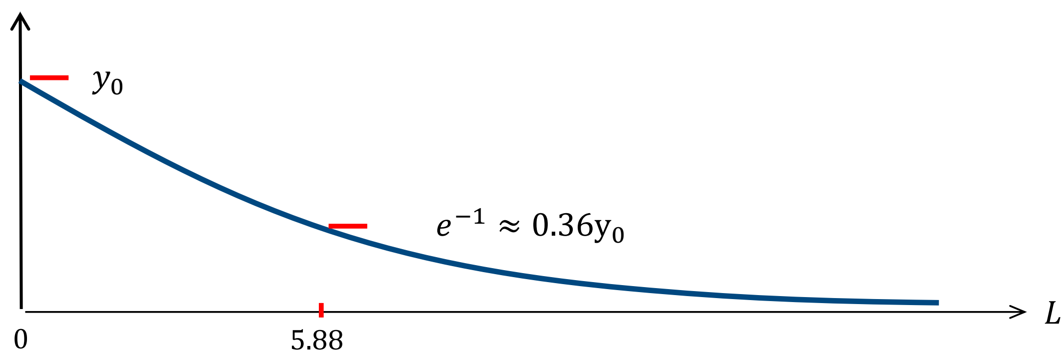
Problem 4d) Sketch the value of y_T as a function of L in cm for $E = 40keV$. (Hint: $1/0.21 = 4.76$.)

Solution:



Problem 4e) Sketch the value of y_T as a function of L in cm for $E = 80keV$. (Hint: $1/0.17 = 5.88$.)

Solution:



Problem 4f) In this problem, assume that the X-ray source energy is at both 40 and 80 keV in equal proportion, so that $y_0 = 1$ for both cases. Then write an expression the measured y_T as a function of L.

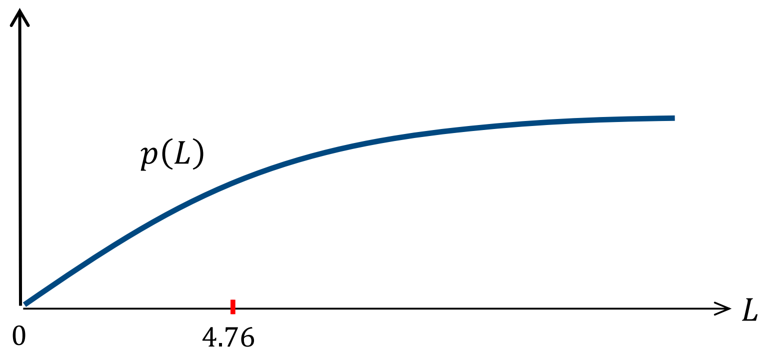
Solution:

$$\begin{aligned} y_T &= \exp\{-\mu_{40}L\} + \exp\{-\mu_{70}L\} \\ &= \exp\{-0.21L\} + \exp\{-0.17L\} \ . \end{aligned}$$

Problem 4g) Assuming that the X-ray source energy is at both 40 and 80 keV as in the previous problem, sketch the value of $p = -\log\left\{\frac{y_T}{y_0}\right\}$ as a function of L.

Solution:

$$p = -\log\left\{\frac{y_T}{y_0}\right\} = -\log\left\{\frac{\exp\{-0.21L\} + \exp\{-0.17L\}}{2}\right\}$$



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Problem 5. (20pt) Color Space Transforms

Consider a color space (r, g, b) with color matching functions given by $r_o, g_o, b_o \in \mathbb{R}^{31}$ and a white point of (x_w, y_w) based on physical red, green, and blue primaries.

Furthermore, let $x_o, y_o, z_o \in \mathbb{R}^{31}$ denote the X, Y, Z color matching functions. And let B be a 3×3 matrix such that

$$\begin{bmatrix} r_0 \\ g_0 \\ b_0 \end{bmatrix} = B \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Problem 5a) Are r_o, g_o, b_o strictly positive? Justify your answer.

Solution: No, the color matching functions for a color space with real primary colors can never be strictly positive.

Problem 5b) Explain how to calculate the white point of (x_w, y_w) from knowledge of B.

Solution: Form a vector of 1's denoted by $\mathbf{1}$, then the X, Y, Z values for the white point are given by

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} = B^{-1}\mathbf{1} \ .$$

Then compute the chromaticity values as

$$(x_w, y_w) = \left(\frac{X_w}{X_w + Y_w + Z_w}, \frac{Y_w}{X_w + Y_w + Z_w} \right) \ .$$

Problem 5c) Explain how to calculate the chromaticity of the red primary, (x_r, y_r) from knowledge of B.

Solution: Compute

$$\begin{bmatrix} X_r \\ Y_r \\ Z_r \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ .$$

Then compute

$$(x_r, y_r) = \left(\frac{X_r}{X_r + Y_r + Z_r}, \frac{Y_r}{X_r + Y_r + Z_r} \right) \ .$$

Problem 5d) Explain how to calculate the (approximate) gamma corrected, $(\tilde{r}, \tilde{g}, \tilde{b})$ values from the values (r, g, b) for $\gamma = 2.4$.

Solution:

$$\tilde{r} = \left(\frac{r}{255}\right)^{1/2.4}$$

$$\tilde{g} = \left(\frac{g}{255}\right)^{1/2.4}$$

$$\tilde{b} = \left(\frac{b}{255}\right)^{1/2.4}$$

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X^*(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx dy$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} du dv$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}} [X(f)]$$

$$\mathrm{rep}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT)$$

$$S(f) = Y\left(e^{j2\pi fT}\right)$$