PURDUE

ECE 63700

Exam #1, February 21, Spring 2025

NAME ____________________________________

PUID ____

Exam instructions:

- A fact sheet is included at the end of this exam for your use.
- You have 50 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

To ensure Gradescope can read your exam:

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

Name/PUID: Key

Problem 1.(15pt) 1D CTFT

Consider the continuous time function $s(t)$ with CTFT given by

$$S(f) = \mathrm{rect}((f - f_c)/f_b) + \mathrm{rect}((f + f_c)/f_b) \ ,$$

where f_c is the center frequency and f_b is the bandwidth with $f_b << f_c$.

Problem 1a) Sketch the function $S(f)$.

Solution:

Problem 1b) Compute the function $s(t)$.

Solution:

$$s(t) = 2\cos(2\pi f_c t) f_b \mathrm{sinc}\,(f_b t)$$

Problem 1c) Sketch the function $s(t)$.

Solution:

Name/PUID: ____________________________________

Problem 2.(20pt) 2D DSFT

Consider the discrete space function $s(m,n) = h(m)h(n)$ where

$$h(n) = \delta(n) - \left[\delta(n-1) + \delta(n+1)\right]/2 \ ,$$

and let $H(\omega)$ be its DTFT.

Problem 2a) Give a simple expression for $H(0)$.

Solution:

$$H(0) = \sum_{n=-\infty}^{\infty} h(n) = 1 - 1 = 0$$

Problem 2b) Calculate a simplified expression for $H(\omega)$.

Solution:

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= 1 - \frac{1}{2}\left[e^{j\omega} + e^{-j\omega}\right] \\ &= 1 - \cos(\omega) \end{aligned}$$

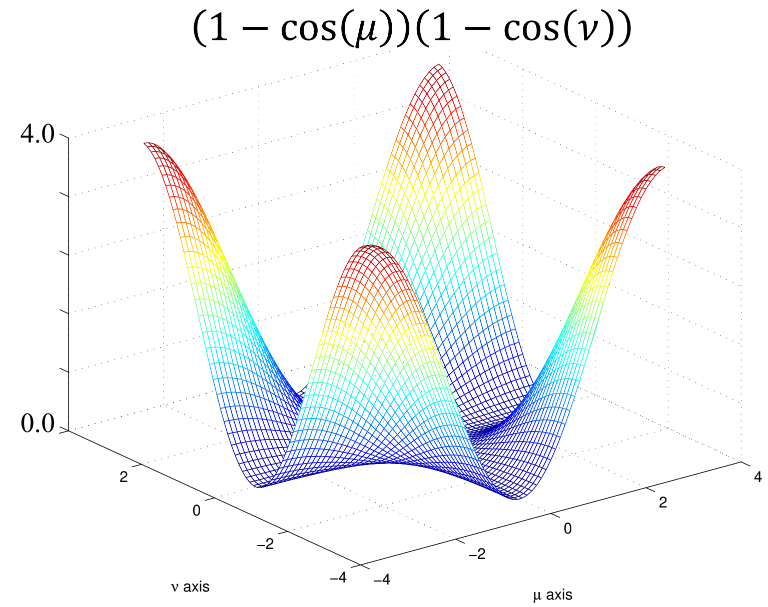
Problem 2c) Calculate $S(\mu,\nu)$, the DSFT of $s(m,n)$.

Solution:

$$\begin{aligned} S(\mu,\nu) &= H(\mu)H(\nu) \\ &= (1-\cos(\mu))(1-\cos(\nu)) \end{aligned}$$

Problem 2d) Sketch the function $|S(\mu,\nu)|$.

Solution:

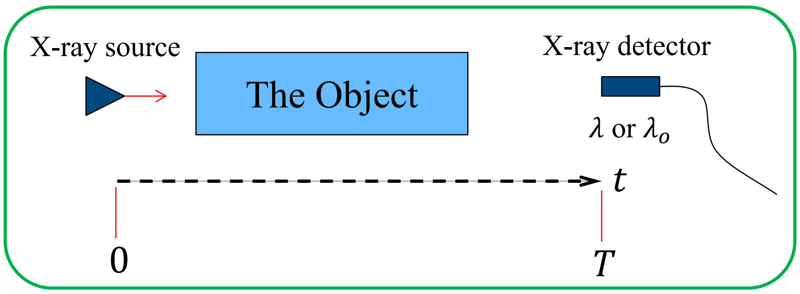

(1 − cos(μ))(1 − cos(ν))
4.0
0.0
2
0
−2
−4
ν axis
−4
−2
0
2
4
μ axis

Name/PUID: ________________________________

Problem 3.(40pt) Tomography

Consider a transmission tomographic imaging system as shown below where λ is the number of photons counted over a fixed time period when the object is present, and λ_o is the number of photons counted over the same fixed time period when the object is removed. Furthermore, assume that the density of the material in units of cm^{-1} is given by $\mu(t)$ where $0 \leq t \leq T$ parameterizes the distance along the path from the source to the detector.

Our goal is to measure the quantity $y = \int_0^T \mu(t)dt$.



Problem 3a) Write an expression for the λ in terms of λ_o and $\mu(t)$.

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Solution:

$$\lambda = \lambda_0 \exp\left\{ -\int_0^T \mu(t)dt \right\}$$

Problem 3b) Write an expression for y in terms of λ and λ_o.

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Solution:

$$\lambda_0 \exp\left\{ -\int_0^T \mu(t)dt \right\} = \lambda$$

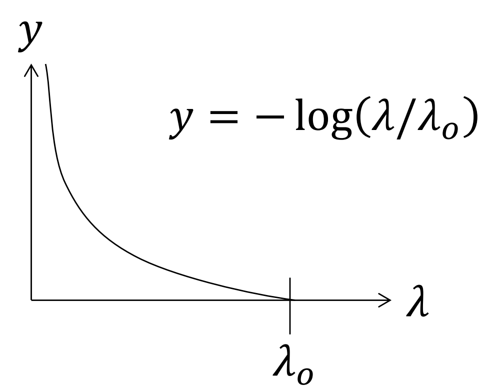
$$\exp\left\{ -\int_0^T \mu(t)dt \right\} = \frac{\lambda}{\lambda_0}$$

$$\int_0^T \mu(t)dt = -\log\left\{ \frac{\lambda}{\lambda_0} \right\}$$

From this, we have that

$$y = \int_0^T \mu(t)dt = -\log\left\{ \frac{\lambda}{\lambda_0} \right\}$$

Problem 3c) Sketch a plot of y versus λ.



Solution:

Problem 3d) For the following sub-problems, assume that

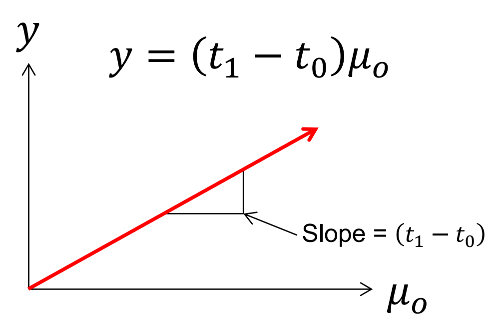
$$\mu(t) = \begin{cases} \mu_o & t_0 \leq t \leq t_1 \\ 0 & \text{otherwise} \end{cases} .$$

Then calculate an expression for y as a function of μ_o, t_0, and t_1.

Solution:

$$y = \int_0^T \mu(t)dt = (t_1 - t_0)\mu_o$$

Problem 3e) Sketch the function y as a function of μ_o for fixed t_0, and t_1.



Solution:

Problem 3f) Calculate the value of λ as a function of μ_o, λ_o, t_0, and t_1.

Solution:

$$y = \int_{t_0}^{t_1} \mu(t)dt = (t_1 - t_0)\mu_o$$

So then we have that

$$\lambda = \lambda_o \exp\{-(t_1 - t_0)\mu_o\}$$

Problem 3g) Sketch the function λ as a function of μ_o for fixed λ_o, t_0, and t_1

Solution:

Problem 3h) What problem might occur if μ_o becomes too large?

Solution: If μ_o becomes too large, then λ might be very small and it may become impossible to measure the signal. This is called "photon starvation".

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Problem 4(30pt) Magnetic Resonance Imaging

Consider an MRI scanner that images in two dimensions, (x, y). So for example, the object being imaged is a sheet in the (x, y) plane.

In this example, assume that the magnetic field strength at each location is given by

$$M(x, y, t) = M_o + x\, G_x(t) + y\, G_y(t) \ ,$$

where M_o is the static magnetic field strength and $x\, G_x(t) + y\, G_y(t)$ is the linear gradient field in the x and y directions. Furthermore, let γ denote the gyromagnetic constant for hydrogen, and let $a(x, y)$ denote the quantity of hydrogen per unit area at location (x, y).

Assume that to initialize the system, the gradient is set to zero, (i.e. $G_x(0) = G_y(0) = 0$) and an RF pulse is transmitted at frequency $\omega_o = \gamma M_o$, so that all the hydrogen atoms in the sheet process. Then the gradient is changed with time.

Problem 4a) Calculate $\omega(x, y, t)$, the frequency of precession of a hydrogen atom at position x, y and time t.

Solution:

$$\omega(x, y, t) = \omega_o + x\, \gamma G_x(t) + y\, \gamma G_y(t)$$

Problem 4b) Calculate $\phi(x, y, t)$, the phase of precession of a hydrogen atom at position (x, y) at time t assuming that $\phi(x, y, 0) = 0$.

Solution:

$$\begin{aligned}
\phi(x, y, t) &= \int_0^t \omega(x, y, \tau) d\tau \\
&= \int_0^t \left[\omega_o + x\, \gamma G_x(\tau) + y\, \gamma G_y(\tau)\right] d\tau \\
&= \omega_o t + x\, \gamma \int_0^t G_x(\tau) d\tau + y\, \gamma \int_0^t G_y(\tau) d\tau \\
&= \omega_o t + x\, k_x(t) + y\, k_y(t)
\end{aligned}$$

Problem 4c) Calculate $r(x, y, t)$, the signal radiated per unit area at position (x, y) at time t.

Solution:

$$\begin{aligned}
r(x, y, t) &= a(x, y)\ \exp\left\{j\omega_o t + jx\, k_x(t) + jy\, k_y(t)\right\} \\
&= a(x, y)\ \exp\left\{j\omega_o t\right\} \exp\left\{jx\, k_x(t) + jy\, k_y(t)\right\}
\end{aligned}$$

Problem 4d) Calculate $r(t)$, the total signal radiated from hydrogen atoms along the entire sheet.

Solution:

$$\begin{aligned} r(t) &= \int_{\mathbb{R}^2} r(x,y,t)dxdy \\ &= \int_{\mathbb{R}^2} a(x,y)\, \exp\{j\omega_o t\} \exp\{jx\, k_x(t) + jy\, k_y(t)\}\, dxdy \\ &= \exp\{j\omega_o t\} \int_{\mathbb{R}^2} a(x,y)\, \exp\{jx\, k_x(t) + jy\, k_y(t)\}\, dxdy \\ &= \exp\{j\omega_o t\}\, A(-k_x(t), -k_y(t))\ , \end{aligned}$$

where

$$A(\mu,\nu) = \int_{\mathbb{R}^2} a(x,y)\, \exp\{-j\mu\, x - j\nu\, y\}\, dxdy\ .$$

Problem 4e) Give expressions for $k_x(t)$ and $k_y(t)$ the spatial frequency (in radians/distance) along the x and y directions in terms of $G_x(t)$ and $G_y(t)$.

Solution:

$$\begin{aligned} k_x(t) &= \gamma \int_0^t G_x(\tau)d\tau \\ k_y(t) &= \gamma \int_0^t G_y(\tau)d\tau\ . \end{aligned}$$

Problem 4f) Describe how the sampled signal should be processed to reconstruct $a(x,y)$?

Solution: Vary the gradient fields, $G_x(t)$ and $G_y(t)$, so that the values of $k_x(t)$ and $k_y(t)$ vary throughout the range of 2D frequencies, $(\mu,\nu) \in [-k_x, k_x] \times [-k_y, k_y]$. Then take the 2D inverse Fourier transform to form $a(x,y)$.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X^*(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(f) &= Y\left(e^{j2\pi f T}\right) \end{aligned}$$