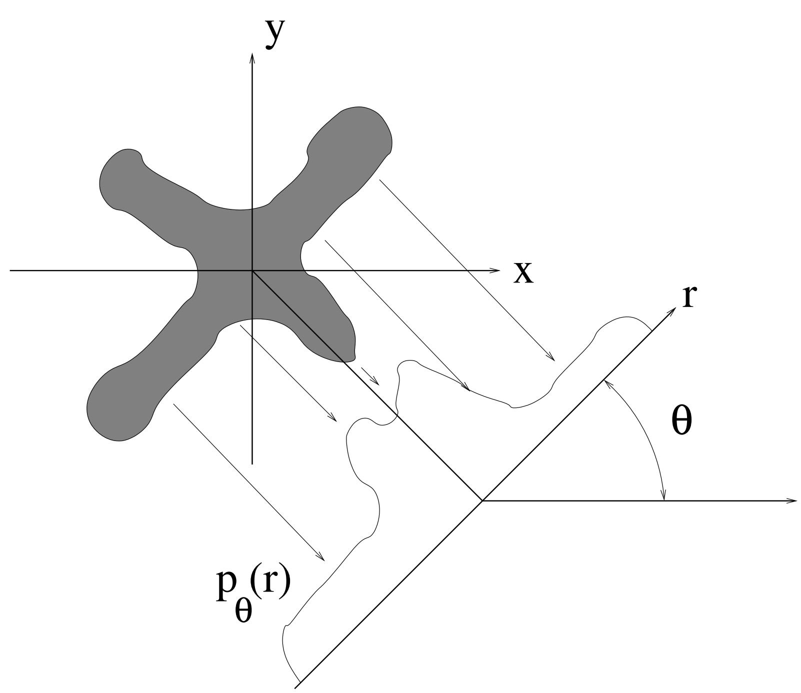
Tomography

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.
  - Projections must be collected at every angle θ and displacement r.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.



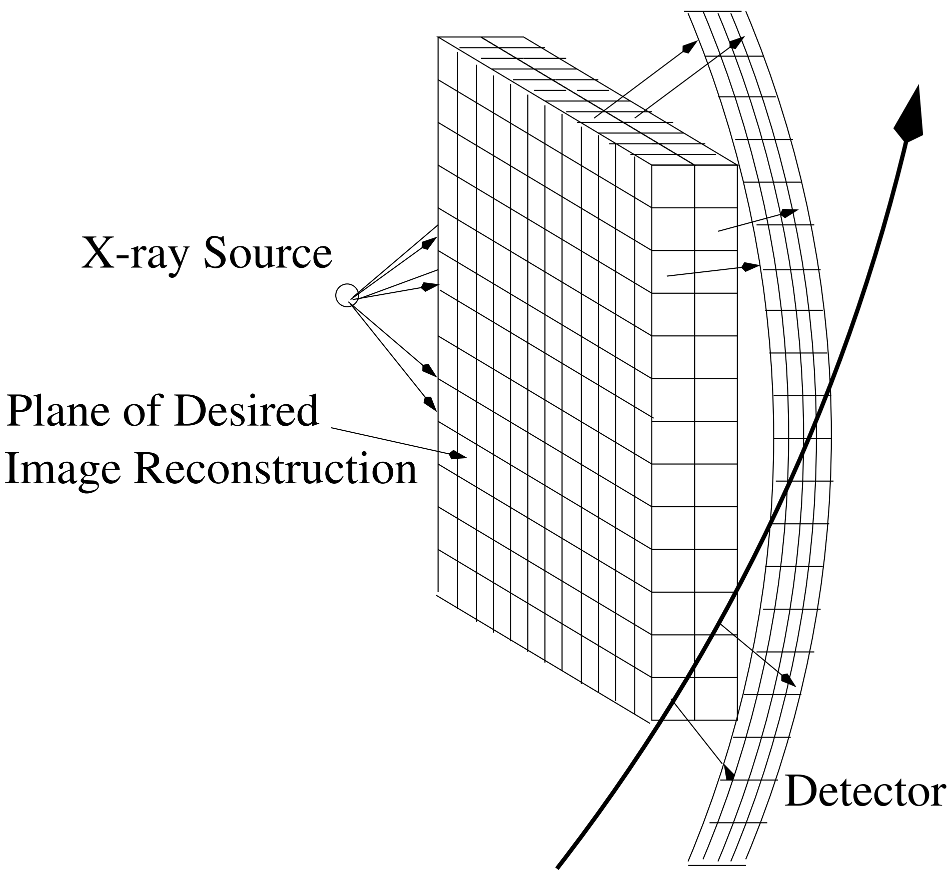
- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

Medical Imaging Modalities

- Anatomical Imaging Modalities
  - Chest X-ray
  - Computed Tomography (CT)
  - Magnetic Resonance Imaging (MRI)
- Functional Imaging Modalities
  - Signal Photon Emission Tomography (SPECT)
  - Positron Emission Tomography (PET)
  - Functional Magnetic Resonance Imaging (fMRI)

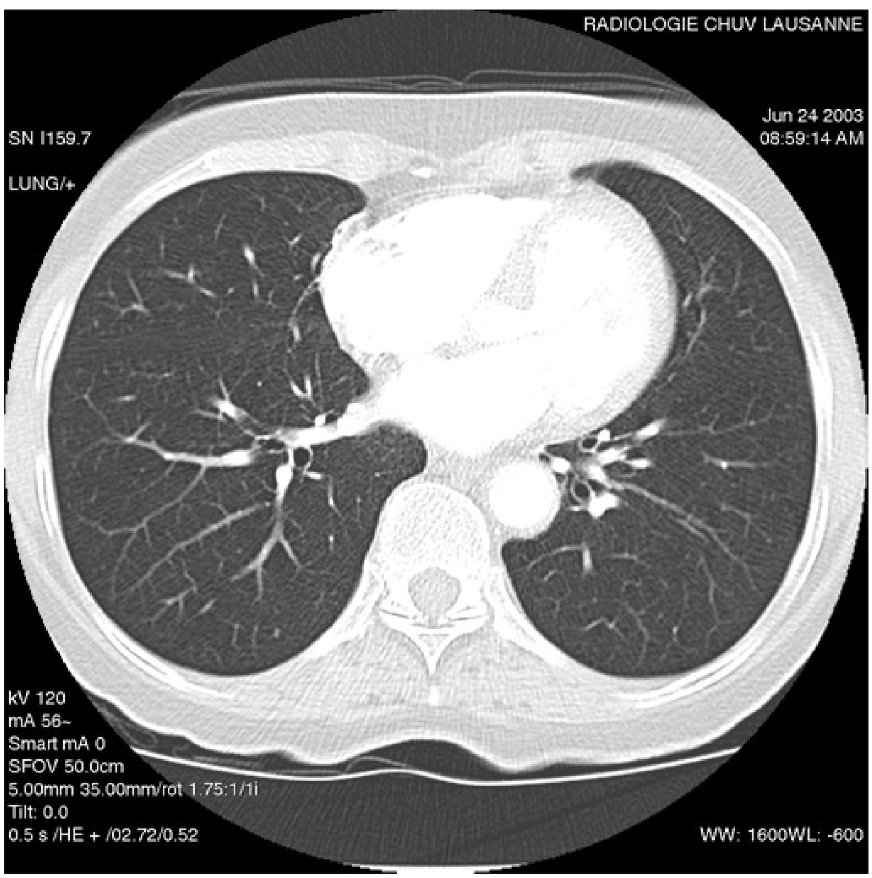
Multislice Helical Scan CT

- Multislice CT has a cone-beam structure

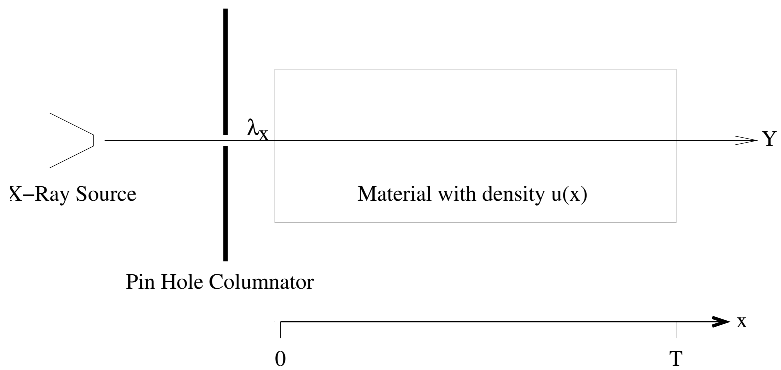


Example: CT Scan

- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan



Photon Attenuation



x - depth into material measured in cm

Y_x - Number of photons at depth x

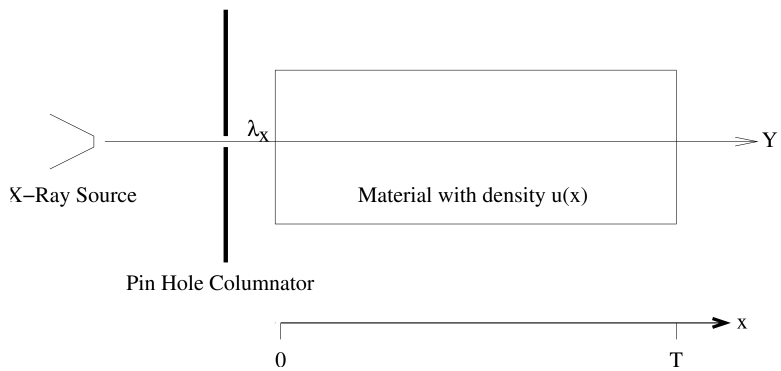
$\lambda_x = E[Y_x]$

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x}\lambda_x^k}{k!} .$$

- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.

Differential Equation for Photon Attenuation



The attenuation of photons obeys the following equation

$$\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x$$

where $\mu(x)$ is the density in units of cm^{-1}.

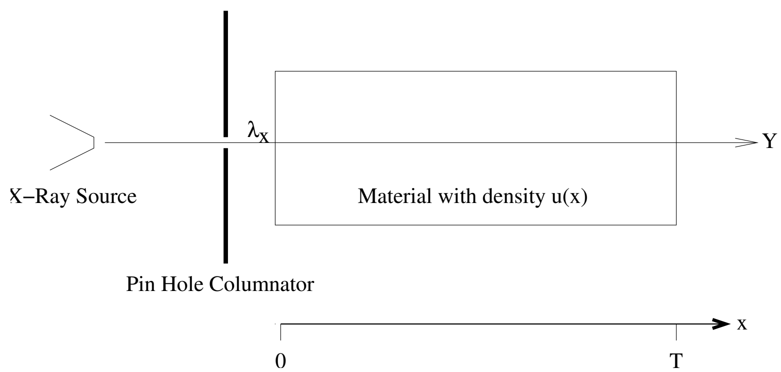
- The solution to this equation is given by

$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t)dt}$$

- So we see that

$$\begin{aligned} \int_0^x \mu(t)dt &= -\log\left(\frac{\lambda_x}{\lambda_0}\right) \\ &\approx -\log\left(\frac{Y_x}{\lambda_0}\right) \end{aligned}$$

Estimate of the Projection Integral



A commonly used estimate of the projection integral is

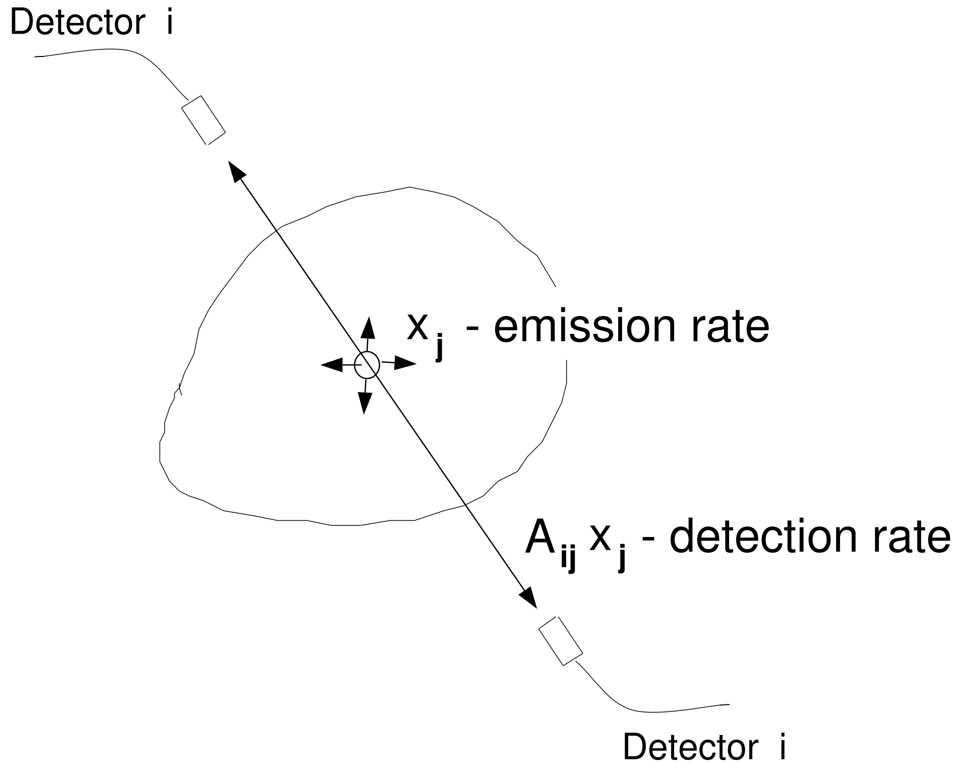
$$\int_0^T \mu(t)dt \cong -\log\left(\frac{Y_T}{\lambda_0}\right)$$

where:

λ_0 is the dosage

Y_T is the photon count at the detector

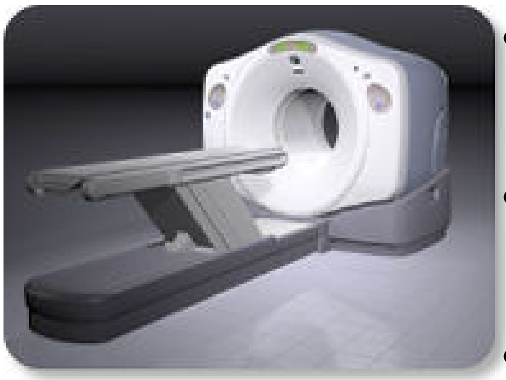
Positron Emission Tomography (PET)



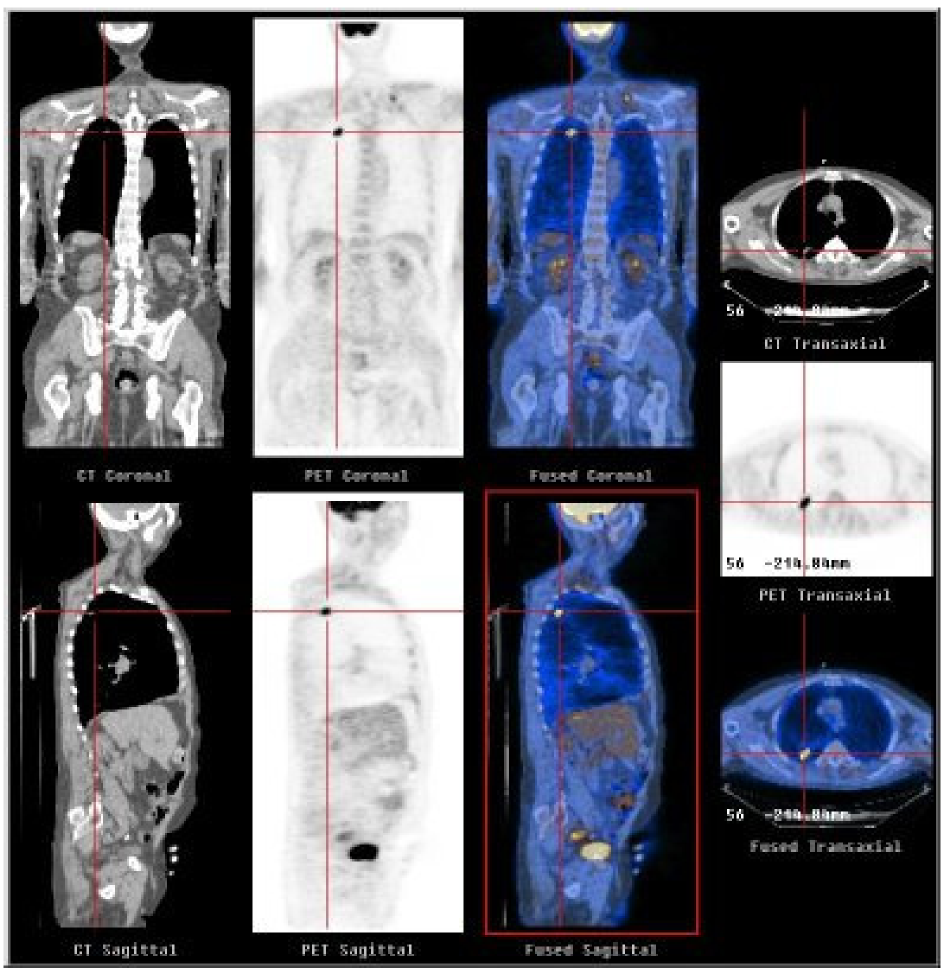
$$E[y_i] = \sum_j A_{ij} x_j$$

- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements

Example: PET/CT Scan



- Generally low space/time resolution
- Little anatomical detail $\Rightarrow$ couple with CT
- Can detect disease



Coordinate Rotation

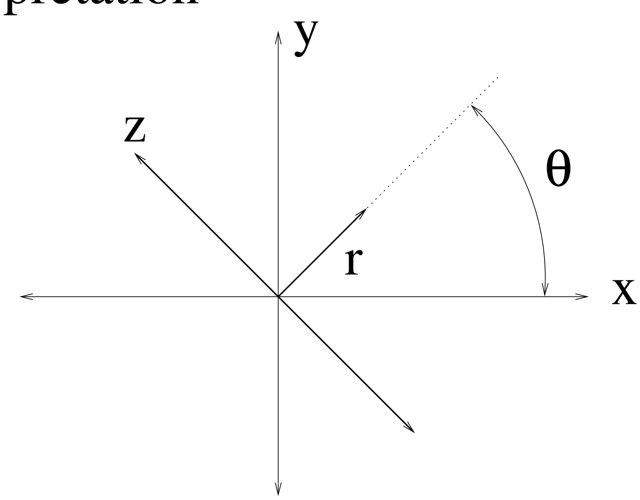
- Define the counter-clockwise rotation matrix

$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Define the new coordinate system (r, z)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$$

- Geometric interpretation

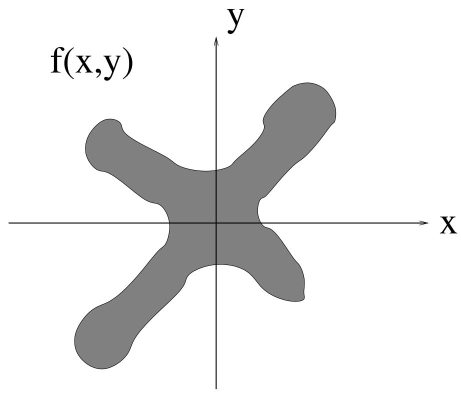


- Inverse transformation

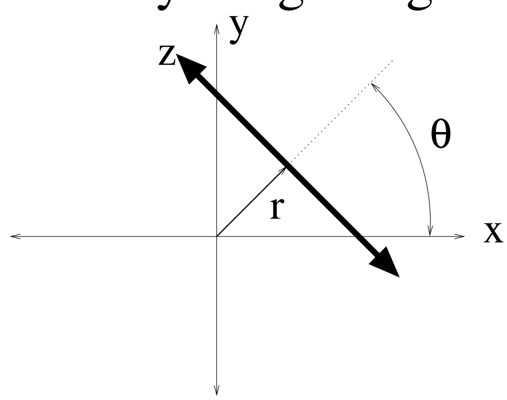
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

Integration Along Projections

- Consider the function $f(x, y)$.



- We compute projections by integrating along z for each r.



- The projection integral for each r and θ is given by

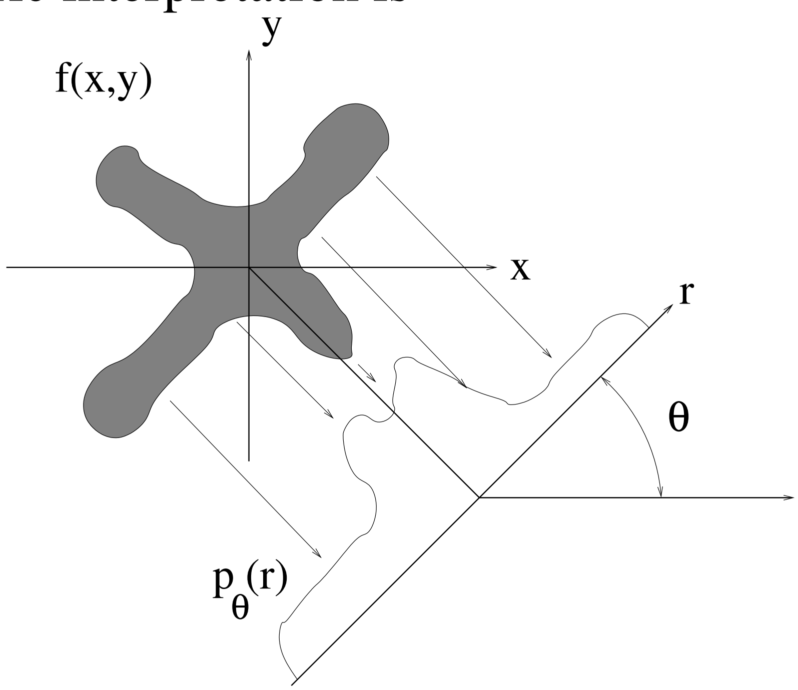
$$
\begin{aligned}
p_\theta(r) &= \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz
\end{aligned}
$$

The Radon Transform

- The Radon transform of the function $f(x,y)$ is defined as

$$p_\theta(r) \;=\; \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz$$

- The geometric interpretation is



Notice that the projection corresponding to $r = 0$ goes through the point $(x,y) = (0,0)$.

The Fourier Slice Theorem

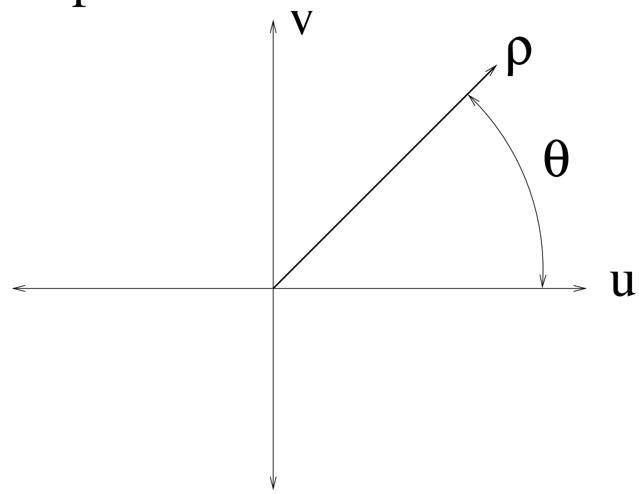
- Let

$$\begin{aligned} P_\theta(\rho) &= CTFT\left\{p_\theta(r)\right\} \\ F(u,v) &= CSFT\left\{f(x,y)\right\} \end{aligned}$$

  Then

$$P_\theta(\rho) = F(\rho\cos(\theta), \rho\sin(\theta))$$

- $P_\theta(\rho)$ is $F(u,v)$ in polar coordinates!



Proof of the Fourier Slice Theorem

- By definition

$$p_\theta(r) = \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz$$

- The CTFT of $p_\theta(r)$ is then given by

$$\begin{aligned}
P_\theta(\rho) &= \int_{-\infty}^{\infty} p_\theta(r) e^{-j2\pi\rho r} dr \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz\right] e^{-j2\pi\rho r} dr \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) e^{-j2\pi\rho r} dz dr
\end{aligned}$$

- We next make the change of variables

$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Notice that the Jacobian is $|\mathbf{A}_\theta| = 1$, and that $r = x\cos(\theta) + y\sin(\theta)$. This results in

$$\begin{aligned}
P_\theta(\rho) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-j2\pi\rho[x\cos(\theta) + y\sin(\theta)]} dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-j2\pi[x\rho\cos(\theta) + y\rho\sin(\theta)]} dx dy \\
&= F(\rho\cos(\theta), \rho\sin(\theta))
\end{aligned}$$

Alternative Proof of the Fourier Slice Theorem

- First let $\theta = 0$, then

$$p_0(r) = \int_{-\infty}^{\infty} f(r,y)\, dy$$

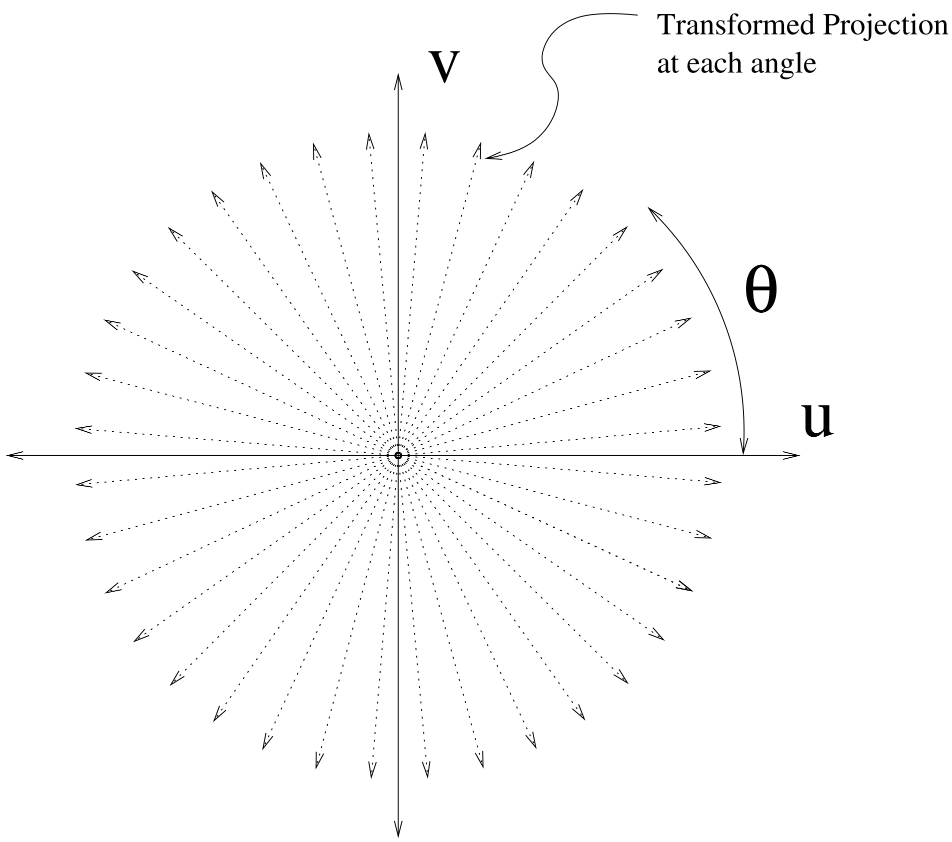
Then

$$\begin{aligned}
P_0(\rho) &= \int_{-\infty}^{\infty} p_0(r) e^{-2\pi j r\rho}\, dr \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(r,y)\, dy \right] e^{-2\pi j r\rho}\, dr \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,y) e^{-2\pi j (r\rho + y0)}\, dr\, dy \\
&= F(\rho, 0)
\end{aligned}$$

- By rotation property of CSFT, it must hold for any θ.

Inverse Radon Transform

- Physical systems measure $p_\theta(r)$.
- From these, we compute $P_\theta(\rho) = CTFT\{p_\theta(r)\}$.



- Next we take an inverse CSFT to form $f(x, y)$.

  Problem: This requires polar to rectagular conversion.

  Solution: Convolution backprojection

Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of $F(u,v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du\,dv = |\rho| d\theta\, d\rho$$

- This results in the expression

$$\begin{aligned}
f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{2\pi j(xu+yv)}\, dudv \\
&= \int_{-\infty}^{\infty}\int_{0}^{\pi} P_\theta(\rho) e^{2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))}\, |\rho| d\theta\, d\rho \\
&= \int_{0}^{\pi} \underbrace{\left[\int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho(x\cos(\theta)+y\sin(\theta))}\, d\rho\right]}_{g_\theta(x\cos(\theta)+y\sin(\theta))} d\theta
\end{aligned}$$

- Then $g(r)$ is given by

$$\begin{aligned}
g_\theta(r) &= \int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho t}\, d\rho \\
&= CTFT^{-1}\left\{|\rho| P_\theta(\rho)\right\} \\
&= h(r) * p_\theta(r)
\end{aligned}$$

where $h(r) = CTFT^{-1}\{|\rho|\}$, and

$$f(x,y) = \int_0^\pi g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

Summary of CBP Algorithm

1. Measure projections $p_\theta(r)$.
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.
3. Back project filtered projections

$$f(x, y) = \int_0^\pi g_\theta \left(x \cos(\theta) + y \sin(\theta) \right) d\theta$$

A Closer Look at the Projection Filter

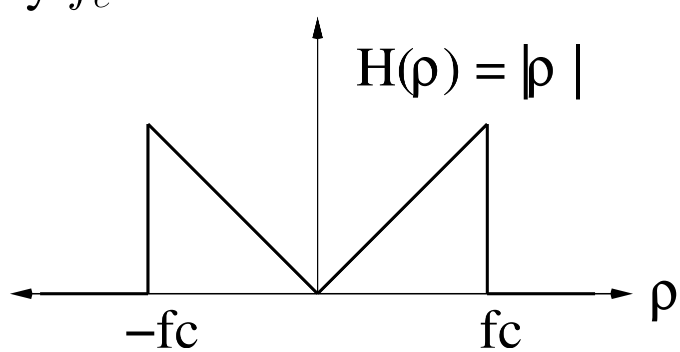
1. At each angle, projections are filtered.

$$g_\theta(r) = h(r) * p_\theta(r)$$

2. The frequency response of the filter is given by

$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to $|\rho| \leq f_c$ for some cut-off frequency f_c.



So

$$\begin{aligned} H(\rho) &= f_c \left[\text{rect}\left(\rho/(2f_c)\right) - \Lambda\left(\rho/f_c\right)\right] \\ h(r) &= f_c^2 \left[2\text{sinc}(r2f_c) - \text{sinc}^2(rf_c)\right] \end{aligned}$$

Discretized Projection Filter

1. Real CT scanners must sample the $\tilde{p}(r) = p(rT_d)$ using a detector array where T_d is the detector pitch.
2. For Nyquist sampling, set the cutoff frequencies to be $f_c = \frac{1}{2T_d}$.

So then we have that

$$\tilde{g}(n) = \tilde{p}(n) * \tilde{h}(n) ,$$

where

$$\begin{aligned}
\tilde{h}(n) &= T_d h(n) \\
&= T_d f_c^2 \left[2\text{sinc}(r2f_c) - \text{sinc}^2(rf_c) \right] \\
&= T_d f_c^2 \left[2\text{sinc}(nT_d 2f_c) - \text{sinc}^2(nT_d f_c) \right] \\
&= \frac{T_d}{4T_d^2} \left[2\text{sinc}(n) - \text{sinc}^2(n/2) \right] \\
&= \frac{1}{T_d} \left[\frac{1}{2}\delta(n) - \frac{1}{4}\text{sinc}^2(n/2) \right]
\end{aligned}$$

So that

$$h(n) = \frac{1}{T_d} \begin{cases} \frac{1}{4} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even} \\ -\frac{1}{(\pi n)^2} & \text{if } n \text{ is odd} \end{cases}$$

A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^\pi b_\theta(x, y)\, d\theta$$

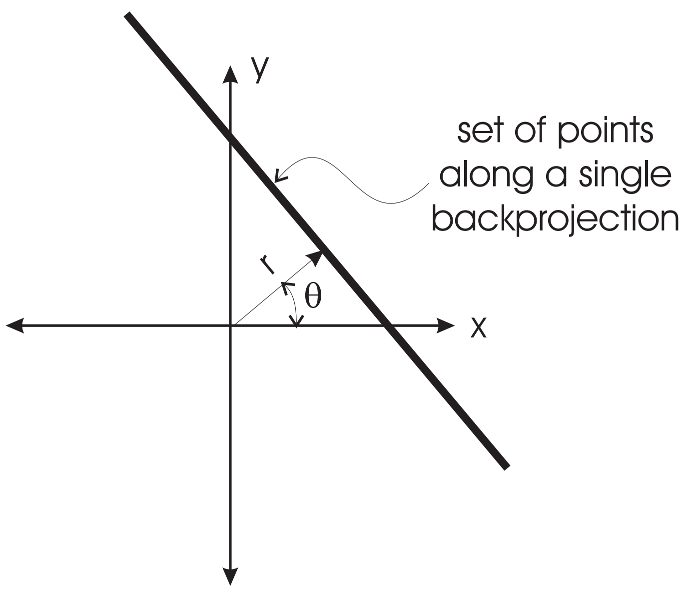
where

$$b_\theta(x, y) = g_\theta(x\cos(\theta) + y\sin(\theta))$$

- Consider the set of points (x, y) such that

$$r = x\cos(\theta) + y\sin(\theta)$$

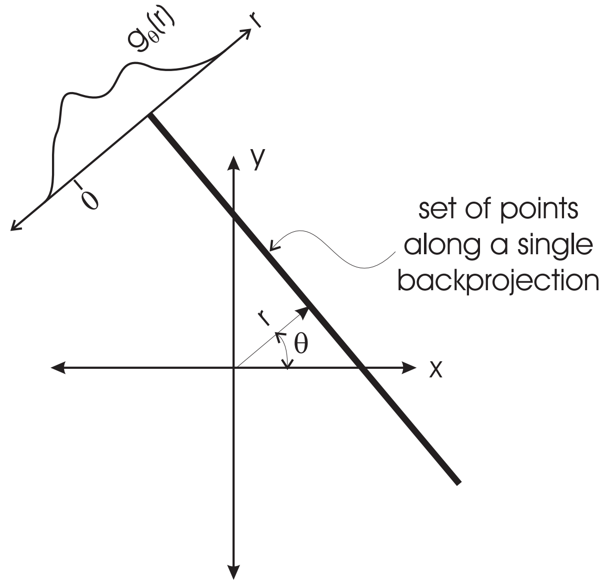
This set looks like



- Along this line $b_\theta(x, y) = g_\theta(r)$.

Back Projection Continued

- For each angle θ back projection is constant along lines of angle θ and takes on value $g_\theta(r)$.



- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to π.

$$f(x,y) = \int_0^\pi b_\theta(x,y)\,d\theta$$

$$\approx \frac{\pi}{M}\sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x,y)$$

- Back projection "smears" values of $g(r)$ back over image, and then adds smeared images for each angle.