2-D Finite Impulse Response (FIR) Filters

- Difference equation

$$y(m,n) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) x(m-k, n-l)$$

- For $N = 2$ - $\circ$ input points; $\times$ output point

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \times & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{array}$$

- Number of multiplies per output point

$$\text{Multiplies} = (2N+1)^2$$

- Transfer function

$$H(z_1, z_2) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) z_1^{-k} z_2^{-l}$$

$$H(e^{j\mu}, e^{j\nu}) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) e^{-j(k\mu + l\nu)}$$

Spatial FIR Smoothing Filtering

- Filter point spread function (PSF) or impulse response: The box, $\boxed{X}$, indicates the center element of the filter.

$$\begin{array}{ccc} 1 & 2 & 1 \\ 2 & \boxed{4} & 2 \\ 1 & 2 & 1 \end{array} \cdot \frac{1}{16}$$

- Apply filter using free boundary condition: Assume that pixels outside the image are 0.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 & 4 & 3 \\ 0 & 0 & 3 & 9 & 12 & 12 & 9 \\ 0 & 0 & 4 & 12 & 16 & 16 & 12 \\ 0 & 0 & 4 & 12 & 16 & 16 & 12 \\ 0 & 0 & 3 & 9 & 12 & 12 & 9 \end{array}}_{\text{Output Image}}$$

PSF for FIR Smoothing Filter

$$
\begin{array}{ccc}
1 & 2 & 1 \\
2 & \boxed{4} & 2 \\
1 & 2 & 1
\end{array}
\cdot \frac{1}{16}
$$

Spatial FIR Horizontal Derivative Filtering

- Filter point spread function (PSF) or impulse response: The box, $\boxed{\text{X}}$, indicates the center element of the filter.

$$\begin{array}{ccc} 2 & 0 & -2 \\ 4 & \boxed{0} & -4 \\ 2 & 0 & -2 \end{array} \cdot \frac{1}{16}$$

- Apply filter using free boundary condition: Assume that pixels outside the image are 0.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 6 & 6 & 0 & 0 & -6 \\ 0 & 0 & 8 & 8 & 0 & 0 & -8 \\ 0 & 0 & 8 & 8 & 0 & 0 & -8 \\ 0 & 0 & 6 & 6 & 0 & 0 & -6 \end{array}}_{\text{Output Image}}$$

PSF of FIR Horizontal Derivative Filter

$$
\begin{array}{ccc}
2 & 0 & -2 \\
4 & \boxed{0} & -4 \\
2 & 0 & -2
\end{array} \cdot \frac{1}{16}
$$

Spatial FIR Vertical Derivative Filtering

- Filter point spread function (PSF) or impulse response: The box, $\boxed{\mathrm{X}}$, indicates the center element of the filter.

$$\begin{array}{ccc} 2 & 4 & 2 \\ 0 & \boxed{0} & 0 \\ -2 & -4 & -2 \end{array} \cdot \frac{1}{16}$$

- Apply filter using free boundary condition: Assume that pixels outside the image are 0.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 8 & 8 & 6 \\ 0 & 0 & 2 & 6 & 8 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -6 & -8 & -8 & -6 \end{array}}_{\text{Output Image}}$$

PSF of FIR Vertical Derivative Filter

$$\begin{array}{ccc} 2 & 4 & 2 \\ 0 & \boxed{0} & 0 \\ -2 & -4 & -2 \end{array} \cdot \frac{1}{16}$$

Example 1: 2-D FIR Filter

- Consider the impulse response $h(m,n) = h_1(m)h_1(n)$ where

$$
\begin{aligned}
4h_1(n) &= (\cdots, 0, 1, 2, 1, 0, \cdots) \\
h_1(n) &= (\delta(n+1) + 2\delta(n) + \delta(n-1))/4
\end{aligned}
$$

Then $h(m,n)$ is a separable function with

$$
16h(m,n) = \begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\
\cdots & 0 & 2 & 4 & 2 & 0 & \cdots \\
\cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

- The DTFT of $h_1(n)$ is

$$
\begin{aligned}
H_1(e^{j\omega}) &= \frac{1}{4}\left(e^{j\omega} + 2 + e^{-j\omega}\right) \\
&= \frac{1}{2}(1 + \cos(\omega))
\end{aligned}
$$

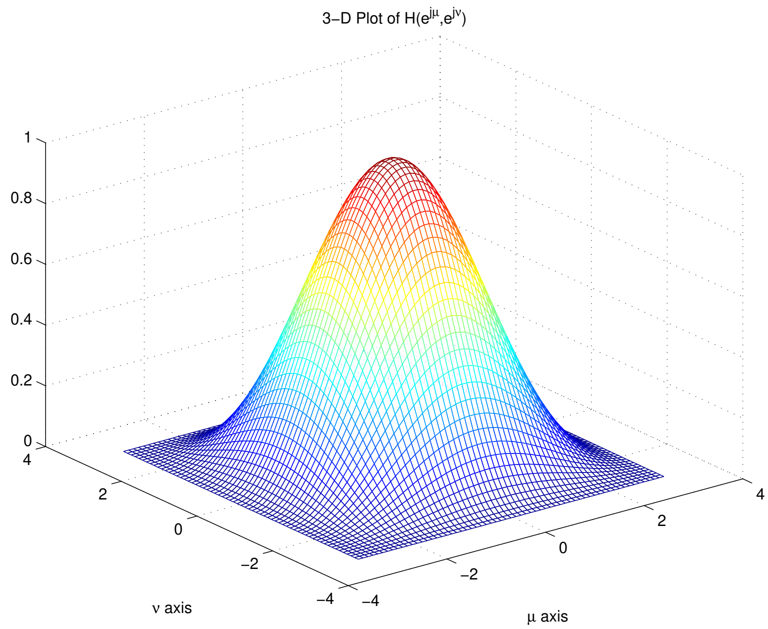
- The DSFT of $h(m,n)$ is

$$
\begin{aligned}
H(e^{j\mu}, e^{j\nu}) &= H_1(e^{j\mu})H_1(e^{j\nu}) \\
&= \frac{1}{4}(1 + \cos(\mu))(1 + \cos(\nu))
\end{aligned}
$$

Example 1: Frequency Response of 2-D FIR Filter

- Plot of frequency response

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4}(1+\cos(\mu))(1+\cos(\nu))$$



- This is a low pass filter with $H(e^{j0}, e^{j0}) = 1$

Example 2: 2-D FIR Filter

- Consider the impulse response $h(m,n) = h_1(m)h_1(n)$ where

$$\begin{aligned} 4h_1(n) &= (\cdots, 0, 1, -2, 1, 0, \cdots) \\ h_1(n) &= (\delta(n+1) - 2\delta(n) + \delta(n-1))/4 \end{aligned}$$

Then $h(m,n)$ is a separable function with

$$16h(m,n) = \begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & 0 & 0 & 0 & 0 & 0 & \\ \cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & 0 & -2 & 4 & -2 & 0 & \cdots \\ \cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\ & 0 & 0 & 0 & 0 & 0 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

- The DTFT of $h_1(n)$ is

$$\begin{aligned} H_1(e^{j\omega}) &= \frac{1}{4}\left(e^{j\omega} - 2 + e^{-j\omega}\right) \\ &= -\frac{1}{2}(1 - \cos(\omega)) \end{aligned}$$

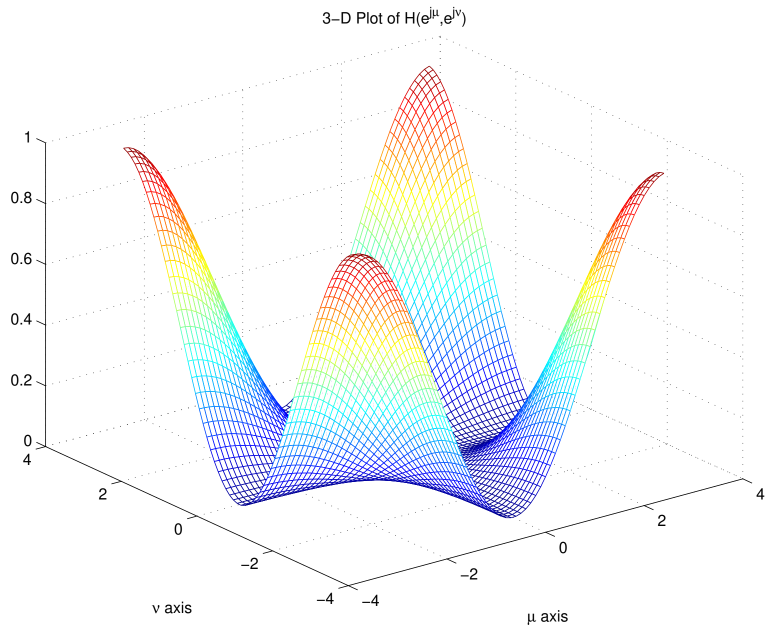
- The DSFT of $h(m,n)$ is

$$\begin{aligned} H(e^{j\mu}, e^{j\nu}) &= H_1(e^{j\mu})H_1(e^{j\nu}) \\ &= \frac{1}{4}(1 - \cos(\mu))(1 - \cos(\nu)) \end{aligned}$$

Example 2: Frequency Response of 2-D FIR Filter

- Plot of frequency response

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4}(1 - \cos(\mu))(1 - \cos(\nu))$$

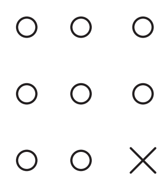


- This is a **high pass** filter with $H(e^{j0}, e^{j0}) = 0$

Ordering of Points in a Plane

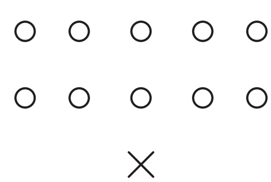
- Recursive filter implementations require the ordering of points in the plane.
- Let $s = (s_1, s_2) \in \mathcal{Z}^2$ and $r = (r_1, r_2) \in \mathcal{Z}^2$.
- Quarter plane - then $s < r$ means:

$$(s_2 < r_2) \text{ and } (s_1 < r_1) \text{ and } s \neq r$$



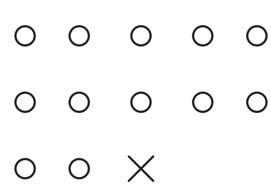
- Symmetric half plane - then $s < r$ means:

$$(s_2 < r_2)$$



- Nonsymmetric half plane - then $s < r$ means:

$$(s_2 < r_2) \text{ or } ((s_2 = r_2) \text{ and } (s_1 < r_1))$$



2-D Infinite Impulse Response (IIR) Filters

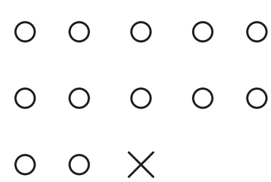
- Difference equation

$$
\begin{aligned}
y(m,n) \ =& \ \sum_{k=-N}^{N} \sum_{l=-N}^{N} b(k,l)x(m-k,n-l) \\
& + \sum_{k=-P}^{P} \sum_{l=1}^{P} a(k,l)y(m-k,n-l) \\
& + \sum_{k=1}^{P} a(k,0)y(m-k,n)
\end{aligned}
$$

Simplified notation

$$
y_s = \sum_{r} b_r x_{s-r} + \sum_{r>(0,0)} a_r y_{s-r}
$$

- For nonsymetric half plane with $N = 0$ and $P = 2$



- Number of multiplies per output point

$$
\text{Multiplies} = \underbrace{(2N+1)^2}_{\text{FIR Part}} + \underbrace{2(P+1)P}_{\text{IIR Part}}
$$

2-D IIR Filter Transfer Functions

- Transfer function in Z-transform domain is

$$H(z_1, z_2) = \frac{\sum_{k=-N}^{N} \sum_{l=-N}^{N} b(k,l) z_1^{-k} z_2^{-l}}{1 - \sum_{k=-P}^{P} \sum_{l=1}^{P} a(k,l) z_1^{-k} z_2^{-l} - \sum_{k=1}^{P} a(k,0) z_1^{-k}}$$

- Transfer function in DSFT domain is

$$H(e^{j\mu}, e^{j\nu}) = \frac{\sum_{k=-N}^{N} \sum_{l=-N}^{N} b(k,l) e^{-j(k\mu + l\nu)}}{1 - \sum_{k=-P}^{P} \sum_{l=1}^{P} a(k,l) e^{-j(k\mu + l\nu)} - \sum_{k=1}^{P} a(k,0) e^{-j(k\mu)}}$$

Example 3: 2-D IIR Filter

- Consider the difference equation

$$y(m,n) = x(m,n) + ay(m-1,n) + ay(m,n-1)$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$\begin{array}{cc} & \circ \\ \circ & \times \end{array}$$

- Taking the Z-transform of the difference equation

$$Y(z_1,z_2) = X(z_1,z_2) + az_1^{-1}Y(z_1,z_2) + az_2^{-1}Y(z_1,z_2)$$

The transfer functions is then

$$\begin{aligned} H(z_1,z_2) &= \frac{Y(z_1,z_2)}{X(z_1,z_2)} = \frac{1}{1-az_1^{-1}-az_2^{-1}} \\ H(e^{j\mu},e^{j\nu}) &= \frac{1}{1-ae^{-j\mu}-ae^{-j\nu}} \end{aligned}$$

Example 3: 2-D IIR Filter in Space Domain

- For $a = 1/2$

$$y(m,n) = x(m,n) + \frac{1}{2}y(m-1,n) + \frac{1}{2}y(m,n-1)$$

- Looks like

$$\begin{array}{cc} & 1/2 \\ 1/2 & \times \end{array}$$

- Apply filter in raster scan order.

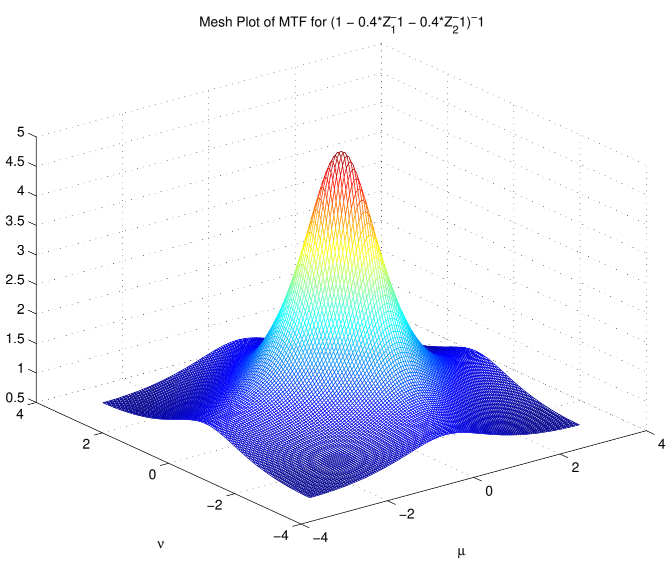
$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 32 & 16 & 8 \\ 0 & 0 & 0 & 32 & 32 & 24 & 16 \\ 0 & 0 & 0 & 16 & 24 & 24 & 20 \\ 0 & 0 & 0 & 8 & 16 & 20 & 20 \end{array}}_{\text{Output Image}}$$

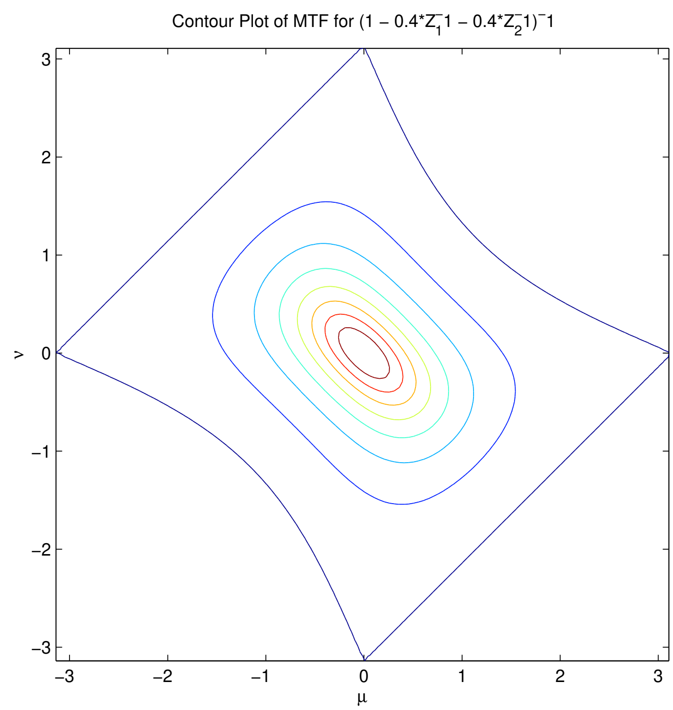
Example 3: Frequency Response of 2-D IIR Filter

- Plot of frequency response

$$H(z_1, z_2) = \frac{1}{1 - az_1^{-1} - az_2^{-1}}$$

for $a = 0.4$.





Example 4: 2-D IIR Filter

- Consider the difference equation

$$
\begin{aligned}
y(m,n) \;=\; & x(m,n) + ay(m-1,n) + ay(m,n-1) \\
& +2ay(m+1,n-1)
\end{aligned}
$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$
\begin{array}{ccc}
 & \circ & \circ \\
\circ & \times &
\end{array}
$$

- The transfer functions is then

$$
\begin{aligned}
H(z_1, z_2) \;=\; & \frac{1}{1 - az_1^{-1} - az_2^{-1} - 2az_1^{+1}z_2^{-1}} \\
H(e^{j\mu}, e^{j\nu}) \;=\; & \frac{1}{1 - ae^{-j\mu} - ae^{-j\mu} - 2ae^{+j\mu - j\nu}}
\end{aligned}
$$

Example 4: 2-D IIR Filter in Space Domain

- For $a = 1/4$

$$\begin{aligned} y(m,n) &= x(m,n) + \frac{1}{4}y(m-1,n) + \frac{1}{4}y(m,n-1) \\ &\quad + \frac{1}{2}y(m+1,n-1) \end{aligned}$$

- Looks like

$$\begin{array}{ccc} & 1/4 & 1/2 \\ 1/4 & \times & \end{array}$$

- Apply filter in raster scan order.

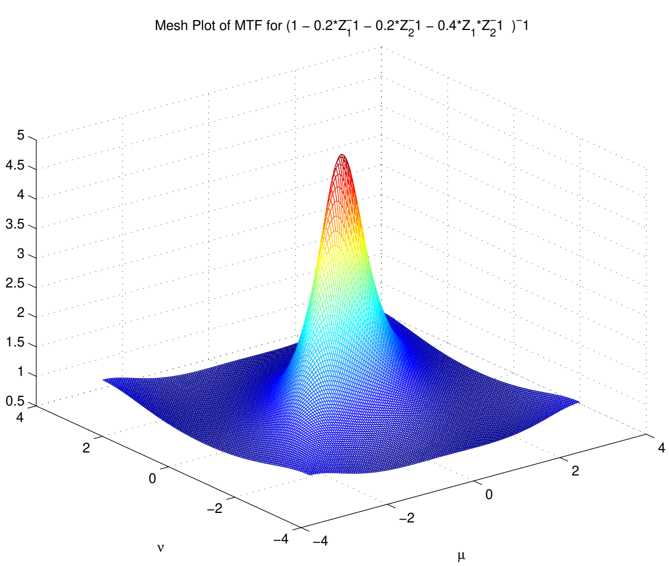
$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 16 & 4 & 1 \\ 0 & 0 & 32 & 32 & 14 & 5 & 1\frac{1}{2} \\ 0 & 16 & 28 & 22 & 11\frac{1}{2} & 4\frac{7}{8} & \frac{45}{32} \end{array}}_{\text{Output Image}}$$

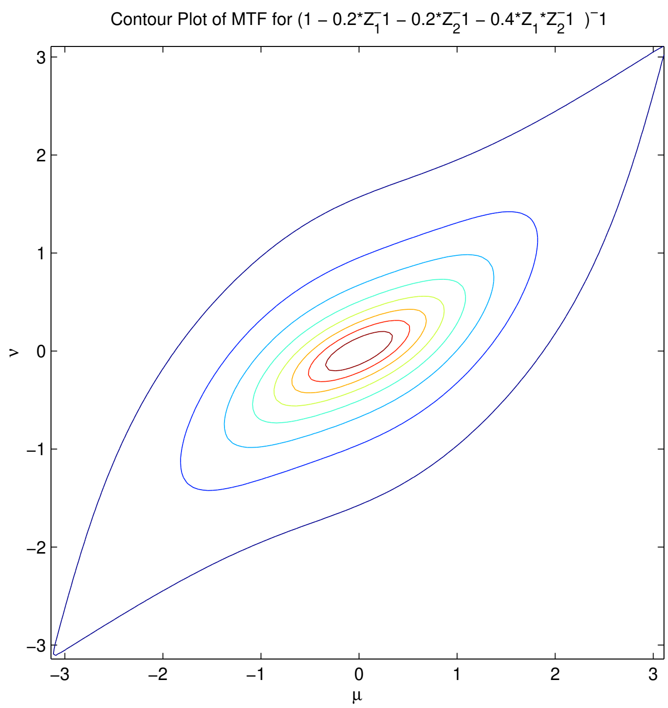
Example 4: Frequency Response of 2-D IIR Filter

- Plot of frequency response

$$H(z_1, z_2) = \frac{1}{1 - a z_1^{-1} - a z_2^{-1} - 2 a z_1 z_2^{-1}}$$

for $a = 0.2$.





- Notice that transfer function has a diagonal orientation.

Example 5: 2-D IIR Filter

- Consider the difference equation

$$\begin{aligned} y(m,n) \; = \; & x(m,n) + ay(m-1,n) + ay(m,n-1) \\ & + ay(m+1,n) + ay(m,n+1) \end{aligned}$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \\ & \circ & \end{array}$$

- Theoretically, the transfer functions is then

$$\begin{aligned} H(z_1, z_1) \; &= \; \frac{1}{1 - az_1^{-1} - az_2^{-1} - az_1 - az_2} \\ H(e^{j\mu}, e^{j\nu}) \; &= \; \frac{1}{1 - ae^{-j\mu} - ae^{-j\mu} - ae^{j\mu} - ae^{j\nu}} \end{aligned}$$

- **THIS DOESN'T WORK**