PURDUE

ECE 63700

Final Exam, April 29, Spring 2024

NAME ______________________________________

PUID ____

Exam instructions:

- A fact sheet is included at the end of this exam for your use.
- You have 120 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

To ensure Gradescope can read your exam:

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

Name/PUID: Key

Problem 1. (35pt) Linear Systems Definition

Let $f : \Re^N \to \Re^M$ be a function.

We say that the function is homogeneous if $\forall \alpha \in \Re$, and $\forall x \in \Re^N$,

$$f(\alpha x) = \alpha f(x) \ .$$

We say that the function is linear if $\forall \alpha \in \Re$, $\forall \beta \in \Re$, $\forall x \in \Re^N$, and $\forall y \in \Re^N$,

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \ .$$

Problem 1a) Consider the function defined by $f(x) = Ax$ where $A \in \Re^{M \times N}$ is a matrix and $x \in \Re^N$ is a column vector. Prove or disprove that f is a homogeneous function.

Solution: Select any $\alpha \in \Re$ and any $x \in \Re^N$, then we have that by the communitive property of multiplication that

$$f(\alpha x) = A(\alpha x) = \alpha A x = \alpha f(x) \ .$$

So the system is homogeneous.

Problem 1b) Consider the function defined by $f(x) = Ax$ where $A \in \Re^{M \times N}$ is a matrix and $x \in \Re^N$ is a column vector. Prove or disprove that f is a linear function.

Solution: Select any $\alpha, \beta \in \Re$ and any $x, y \in \Re^N$, then we have that by the distributive property of matrix multiplication and the commutative property of multiplication that

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) = A(\alpha x) + A(\beta y) = \alpha A(x) + \beta A(y) = \alpha f(x) + \beta f(y) \ .$$

So the system is linear.

Problem 1c) Consider the function $g : \Re^N \to \Re^M$ defined by $g(x) = \text{median}(x)$ where N is odd and $M = 1$. Prove or disprove that g is a homogeneous function.

Solution: Select any $\alpha \in \Re$ and any $x = [x_0, \cdots, x_{N-1}] \in \Re^N$, then we have that

$$\begin{aligned} f(\alpha x) &= \text{median}(\alpha [x_0, \cdots, x_{N-1}]) \\ &= \text{median}([\alpha x_0, \cdots, \alpha x_{N-1}]) \\ &= \alpha x_{k^*} = \alpha \text{median}([x_0, \cdots, x_{N-1}]) \\ &= \alpha f(x) \ , \end{aligned}$$

where k^* is the element in the list what has the median rank. Notice that k^* is the same for both lists $[x_0, \cdots, x_{N-1}]$ and $[\alpha x_0, \cdots, \alpha x_{N-1}]$ because multiplication by α does not changing the ordering of the elements. So the system is homogeneous.

Problem 1d) Consider the function $g : \Re^N \to \Re^M$ defined by $g(x) = \text{median}(x)$ where N is odd and $M = 1$. Prove or disprove that g is a linear function.

Solution: The system is not linear, which means we must prove that

$$\neg \left[\forall \alpha, \beta \in \Re, \forall x, y \in \Re^N, f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)\right]$$
$$\iff$$
$$\exists \alpha, \beta \in \Re, \exists x, y \in \Re^N \text{ such that } f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$$

So to prove this, we must find values such that the equality does not hold. We choose $\alpha = \beta = 1$ and $x = [1, 0, 0]$ and $y = [0, 1, 0]$. Then we have that

$$f(\alpha x + \beta y) = f([1, 1, 0]) = 1 \ ,$$

but

$$\alpha f(x) + \beta f(y) = f([1, 0, 0]) + f([0, 1, 0]) = 0 + 0 = 0 \ .$$

So therefore, we have that

$$f(\alpha x + \beta y) = 1 \neq 0 = \alpha f(x) + \beta f(y) \ .$$

So the system is nonlinear, i.e., it is not linear.

Problem 1e) Let $f : \Re^N \to \Re^M$ be a linear function, and let $x = [0, \cdots, x_i, \cdots, 0]$ where $x_i \in \Re$. Prove that $\forall i \in [0, \cdots, N-1]$, $\exists e_i \in \Re^M$ such that $\forall x_i \in \Re$,

$$f(x) = x_i e_i \ .$$

Solution:

First, for any $i \in [0, \cdots, N-1]$ define

$$e_i = f(\delta^{(i)}) \ .$$

where $\delta^{(i)} = [0, \cdots, \underset{i^{th}\text{location}}{1}, \cdots, 0]$. Then for any $i \in [0, \cdots, N-1]$, we have that

$$\begin{aligned} f(x) &= f(x_i \delta^{(i)}) \\ &= x_i f(\delta^{(i)}) \\ &= x_i e_i \end{aligned}$$

So the result is proved.

Problem 1f) Let $f : \Re^N \to \Re^M$ be a linear function, and let $x = [x_0, \cdots, x_{N-1}]$ where $x \in \Re^N$. Then prove that $\forall i \in [0, \cdots, N-1]$, $\exists e_i \in \Re^M$ such that $\forall x \in \Re^N$,

$$f(x) = \sum_{i=0}^{N-1} x_i e_i \ .$$

Solution: First notice that

$$x = \sum_{i=0}^{N-1} x_i \delta^{(i)} \ .$$

So then we have that

$$f(x) = f\left(\sum_{i=0}^{N-1} x_i \delta^{(i)} \right) = \sum_{i=0}^{N-1} x_i f\left(\delta^{(i)}\right) = \sum_{i=0}^{N-1} x_i e_i$$

So the result is proved.

Problem 1g) Use the solution of part f) above to prove that if $f : \Re^N \to \Re^M$ be a linear function, then $\exists A \in \Re^{M \times N}$ such that

$$f(x) = Ax \ .$$

Solution: Define the matrix A such that

$$A = \left[\begin{array}{ccc} e_0 & \cdots & e_{N-1} \end{array} \right] \ .$$

Then we have that

$$Ax = \sum_{i=0}^{N-1} x_i e_i = f(x) \ .$$

So the result is proved.

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Problem 2. (30pt) Least Squares vs MMSE Estimators

Let (Y_k, X_k) for $k = 0, \cdots, K-1$ be training data pairs that are i.i.d. zero mean and Gaussian such that $Y_k \in \Re^N$ and $X_k \in \Re$.

Our goal is to design an estimator, $\hat{X}_k = f_\theta(Y_k) = \theta^t Y_k$.

Problem 2a) Use the training data to formulate an expression for a loss function $L(\theta)$ that represents the total squared estimation error.

Solution:

$$L(\theta) = \sum_{k=0}^{K-1} \|X_k - \theta^t Y_k\|^2$$

Problem 2b) Is the loss function, $L(\theta)$, a random variable or a number? Why?

Solution: The loss function $L(\theta)$ is a random variable because X_k and Y_k are both random.

Problem 2c) Calculate a value of $\hat{\theta}$ that minimizes $L(\theta)$.
Is $\hat{\theta}$ a random variable or a number?

Solution:

$$\begin{aligned}
0 &= \nabla_\theta L(\theta) \\
&= \nabla_\theta \sum_{k=0}^{K-1} \|X_k - \theta^t Y_k\|^2 \\
&= \sum_{k=0}^{K-1} \nabla_\theta \|X_k - \theta^t Y_k\|^2 \\
&= \sum_{k=0}^{K-1} \left\{ 2(X_k - \theta^t Y_k)(-Y_k^t) \right\} \\
&= -\sum_{k=0}^{K-1} \left\{ X_k Y_k^t \right\} + \theta^t \sum_{k=0}^{K-1} \left\{ Y_k Y_k^t \right\}
\end{aligned}$$

So if we then define

$$\hat{b} = \frac{1}{K}\sum_{k=0}^{K-1}\left\{X_k Y_k^t\right\}$$
$$\hat{R} = \frac{1}{K}\sum_{k=0}^{K-1}\left\{Y_k Y_k^t\right\} \ ,$$

then we have that

$$\hat{\theta} = \hat{R}^{-1}\hat{b} \ .$$

Notice that $\hat{\theta}$ is a random variable because it is a function of the random variables X_k and Y_k.

Problem 2d) Define $\bar{L}(\theta) = E[L(\theta)]$. Calculate the value θ^* that minimizes $\bar{L}(\theta)$.

Solution: Notice that

$$\begin{aligned}
\bar{L}(\theta) &= E\left[\sum_{k=0}^{K-1}\|X_k - \theta^t Y_k\|^2\right] \\
&= \sum_{k=0}^{K-1} E\left[\|X_k - \theta^t Y_k\|^2\right] \\
&= \sum_{k=0}^{K-1} E\left[X_k^2 - 2\theta^t X_k Y_k + \theta^t Y_k Y_k^t \theta\right] \\
&= \sum_{k=0}^{K-1} a - 2\theta^t b + \theta^t R\theta \\
&= K\left(a - 2\theta^t b + \theta^t R\theta\right)
\end{aligned}$$

So then we have that

$$0 = \frac{1}{2K}\nabla_\theta = -b^t\theta + R\theta$$

So we have that

$$\theta^* = R^{-1}b \ .$$

Problem 2e) Is $\bar{L}(\theta)$ a random variable or a number? Why?
Is θ^* a random variable or a number? Why?

Solution: The loss function $\bar{L}(\theta)$ is a number because it is the expectation of something. Since θ^* is the minimum of a deterministic function, it is also deterministic.

Problem 2f) Consider the following two estimators:

$$X^* = f_{\theta^*}(Y_k) = [\theta^*]^t Y_k$$
$$\hat{X} = f_{\hat{\theta}}(Y_k) = [\hat{\theta}]^t Y_k$$

What names do we give these estimators?
What are the advantages and disadvantages of each estimator?

Solution: X^* is the minimum mean squared estimator. $\hat{X}$ is the least squares estimate.

The minimum mean squared estimator is the estimator that minimizes the square error, but it is difficult to know since it requires that on know the joint distribution of X, Y.

The least squares estimate is more practical because you can estimate its parameters from training examples. But the estimates are not as good as the MMSE estimator.

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Problem 3. (25pt) Sampling a Sinc Function

Consider the following system where $s(t)$ is a continuous time signal that is band limited to a maximum frequency of f_c, and $z(n)$ and $y(n)$ are discrete time signals.

$$x(n) = s(nT)$$

$$z(n) = \begin{cases} x(n/L) & \text{if } n \bmod L = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$y(n) = \sum_m h(n-m)z(m)$$

Problem 3a) What is the upper bound T_{max} such that $T < T_{max}$ guarantees perfect reconstruction? What is $1/T_{max}$ called?

Solution: The upper bound is $T_{max} < \frac{1}{2f_c}$. $1/T_{max} = 2f_c$ is called the Nyquist rate or frequency.

Problem 3b) Find an expression for the DTFT, $X(e^{j\omega})$, in terms of the CTFT, $S(f)$.

Solution:

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

Problem 3c) Assuming that $T < T_{max}$, then find an expression for $S(f)$ in terms of $X(e^{j\omega})$.

Solution: In this case, there is no aliasing, so we have that

$$S(f) = X(e^{j2\pi Tf})\,\mathrm{rect}(Tf) \ .$$

Problem 3d) Assuming that $T < T_{max}$, then find an expression for $s(t)$ in terms of $x(n)$.

Solution:

$$s(t) = \sum_{n=-\infty}^{\infty} \operatorname{sinc}(t/T)\; x(n)$$

Problem 3e) Determine a filter, $h(n)$ so that $y(n) = s(nT/2)$.

Solution:

$$h(n) = \operatorname{sinc}(n/2)$$

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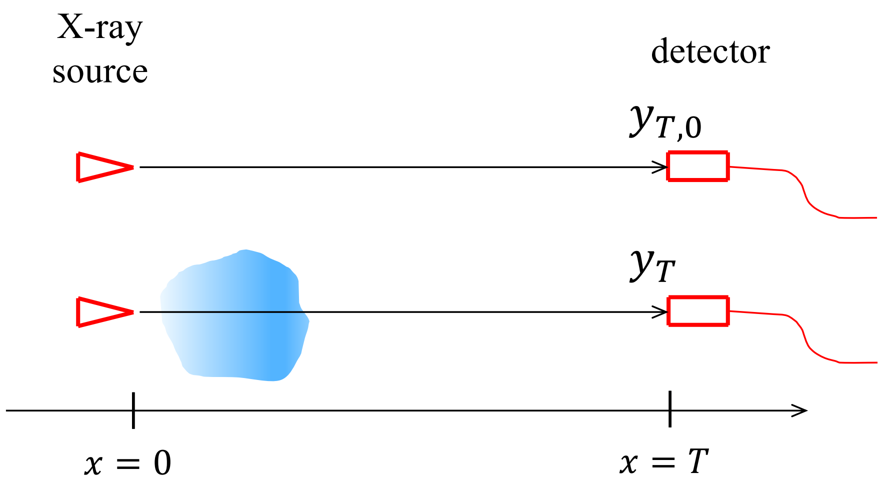
Problem 4. (25pt) Measuring Material Path Integrals

Consider the X-ray transmission imaging system pictured below with a mono-energetic source at 70 keV.



We know that the photon flux, $y(x)$, as a function of depth, x, (units of cm) obeys the following equation

$$\frac{dy(x)}{dx} = -\mu(x)y(x) \ ,$$

where $\mu(x)$ (units of cm^{-1}) is the X-ray density of the material at location x.

Also, $y_{T,0} = y(T)$ when the object is removed (i.e., the blank scan), and $y_T = y(T)$ when the object is in place (i.e., the object scan).

Furthermore, assume that you are imaging a solid material, such as for example steel. Then let $I(x)$ be an indicator function so that $I(x) = 1$ indicates that the steel is present at location x, and $I(x) = 0$ indicates that it is absent. In this case, we have that

$$\mu(x) = \mu_o I(x) \ ,$$

where μ_o is the optical density of steal at 70 keV.

Problem 4a) Derive a formula for the wavelength of the X-rays, λ, in units of cm?
As the energy increases what happens to the wavelength of the X-rays?
(Specify any constants that are required in terms of their physical interpretation or meaning.)

Solution: Then energy of a photon is given by $E = h\nu$, where h is Plank's constant and ν is the frequency. In addition, the frequency is given by $\nu = \frac{C}{\lambda}$, where C is the speed-of-light in free space. So then we have that

$$E = \frac{hC}{\lambda} \ .$$

Problem 4b) Find a solution to the differential equation, $y(x)$, which meets the boundary condition that $y(0) = y_0$.

Solution:

$$y(t) = y_o \exp\left\{ - \int_0^t \mu(\tau) d\tau \right\}$$

Problem 4c) Derive an expression for $\int_0^T \mu(\tau)d\tau$ in terms of $y_{T,0}$ and y_T.

Solution:

$$y_T = y_o \exp\left\{ - \int_0^T \mu(\tau) d\tau \right\}$$
$$y_{T,0} = y_o \exp\left\{ - \int_0^T 0 d\tau \right\} = y_o$$

So then we have that

$$\int_0^T \mu(\tau)d\tau = -\log \exp\left\{ - \int_0^T \mu(\tau) d\tau \right\} = -\log \frac{y_T}{y_{T,0}}$$

Problem 4d) Derive an expression for the path length through the steel, $\int_0^T I(\tau)d\tau$, in terms of $y_{T,0}$ and y_T.

Solution:

$$\int_0^T I(\tau)d\tau = \frac{1}{\mu_0}\int_0^T \mu_0 I(\tau)d\tau = \frac{1}{\mu_0}\int_0^T \mu_o(\tau)d\tau = -\frac{1}{\mu_0}\log\left(\frac{y_T}{y_{T,0}}\right)$$

Problem 4e) Your goal is to form a 3D reconstruction of a additively manufactured steel part from transmission tomographic projections at exactly 70 keV.
Explain in words the preprocessing steps that are required before reconstruction.

Solution: You need to make a blank scan, y_0, and a scan of the object, y. Then you form a sinogram with the form

$$p = -\frac{1}{\mu_0} \log\left(\frac{y_T}{y_{T,0}}\right) .$$

Then do a tomographic reconstruction from p using, for example, filtered back projection. The final reconstruction would be approximately 1 in the steel region, and approximately 0 in the air.

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Problem 5. (25pt) White Noise Driven System

In an experiment, you collect a sequence of samples, Y_n for $n = 0, \cdots, N-1$.

Your goal in this problem is to design an algorithm for generating new synthetic samples Z_n that have the same distribution as Y_n. To do this, you will model Y_n as being a portion of a zero-mean stationary Gaussian random process.

Furthermore, let $\hat{Y}_n = Z_n\theta$ by an estimator of Y_n given the past P samples, $Z_n = [Y_{n-1}, \cdots, Y_{n-P}]$.

Problem 5a) Write out the equation for $\hat{\theta}$, the least-squares estimate of θ.

Solution:

$$\hat{b} = \frac{1}{N-P}\sum_{n=P}^{N-1} Y_n Z_n^t$$

$$\hat{R} = \frac{1}{N-P}\sum_{n=P}^{N-1} Z_n^t Z_n$$

Then the least squares solution is given by

$$\hat{\theta} = \hat{R}^{-1}\hat{b}$$

Problem 5b) Assume that $\hat{Y}_n = Z_n\hat{\theta}$ is the minimum mean squared error estimate of Y_n given Y_r for $r < n$. Then calculate the autocorrelation function for $\epsilon_n = Y_n - \hat{Y}_n$.
(Hint: Use a known property of ϵ_n.)

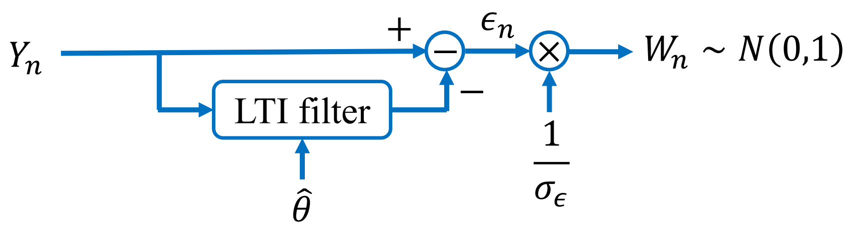
Solution: We know that if $\hat{Y}_n$ is the MMSE estimate, then the prediction errors are white. So

$$R_\epsilon(n) = \sigma_\epsilon^2 \delta(n)$$

for some variance σ_ϵ^2.

Problem 5c) Draw a flow diagram for the "analysis" system that starts with Y_n and ends with i.i.d. Gaussian random variables, W_n.

Solution:



Problem 5d) Write the equations for a "synthesis system" that reverses the analysis system above, has an input of the i.i.d. Gaussian random variables W_n, and an output of Z_n.

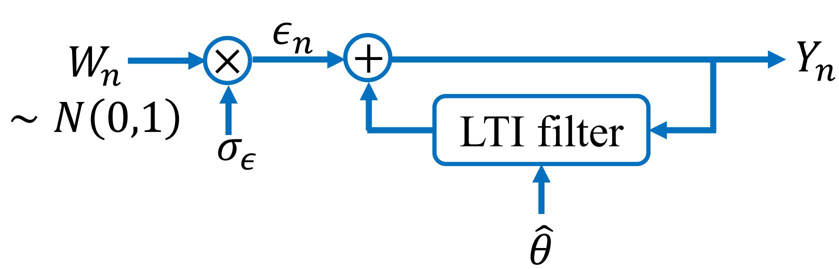
Solution:

$$\epsilon_n \leftarrow \sigma_\epsilon W_n$$

$$Y_n \leftarrow \sum_{i=1}^{P} Y_{n-i}\hat{\theta}_{i-1}$$

Problem 5e) Draw the flow diagram for the "synthesis system" and label the inputs, outputs, and intermediate signals.

Solution:



Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\text{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\text{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}}[X(f)]$$

$$\text{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$