

Wiener-Khintchine Theorem

For a well behaved stationary random process the power spectrum is equal to the Fourier transform of the autocorrelation function.

$$S_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}$$

Sloppy proof:

$$\begin{aligned}
 S_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} E[|X_N(e^{j\omega})|^2] \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} E \left[\left(\sum_{n=-N}^N x(n) e^{-j\omega n} \right) \left(\sum_{k=-N}^N x(k) e^{-j\omega k} \right)^* \right] \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} E \left[\sum_{n=-N}^N \sum_{k=-N}^N x(n) x(k) e^{-j\omega(n-k)} \right] \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \sum_{k=-N}^N E[x(n) x(k)] e^{-j\omega(n-k)} \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \sum_{k=-N}^N R_x(n-k) e^{-j\omega(n-k)} \\
 &= \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \sum_{k=-M}^M R_x(n-k) e^{-j\omega(n-k)} \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \lim_{M \rightarrow \infty} \sum_{k=-M}^M R_x(n-k) e^{-j\omega(n-k)} \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \left(\sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k} \right) \\
 &= \left(\sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k} \right) \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N 1 \\
 &= \sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}
 \end{aligned}$$