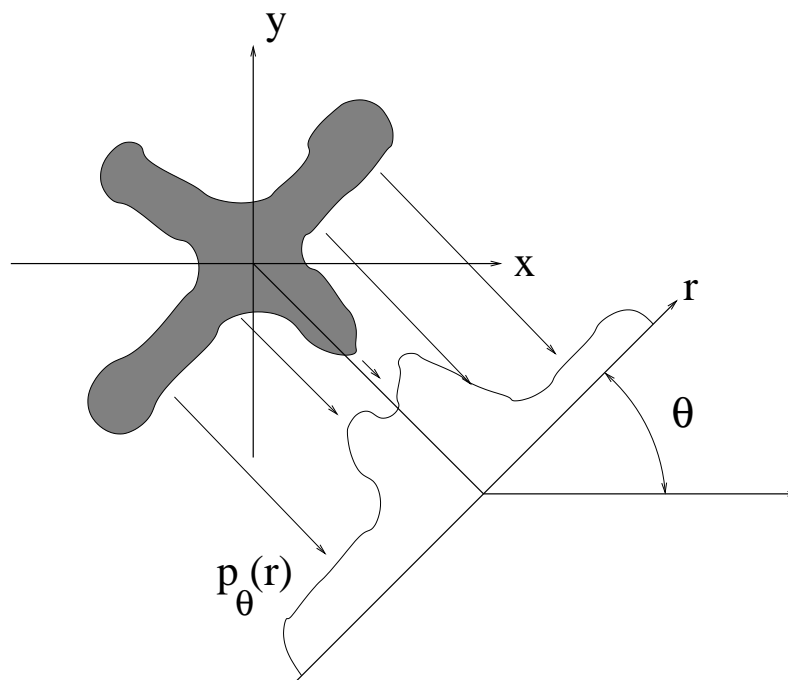


# Tomography

- Many medical imaging systems can only measure projections through an object with density  $f(x, y)$ .
  - Projections must be collected at every angle  $\theta$  and displacement  $r$ .
  - Forward projections  $p_\theta(r)$  are known as a Radon transform.



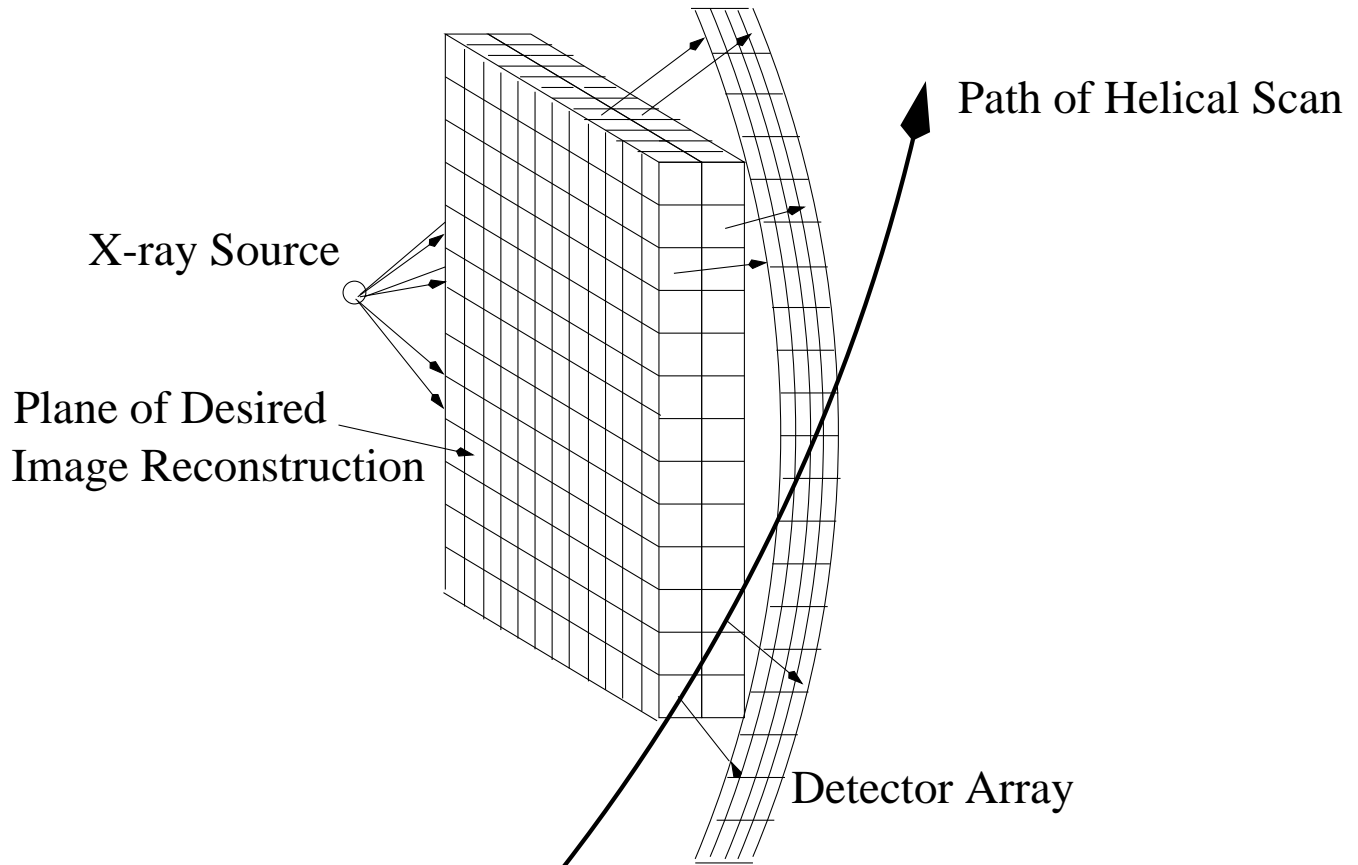
- Objective: reverse this process to form the original image  $f(x, y)$ .
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

## **Medical Imaging Modalities**

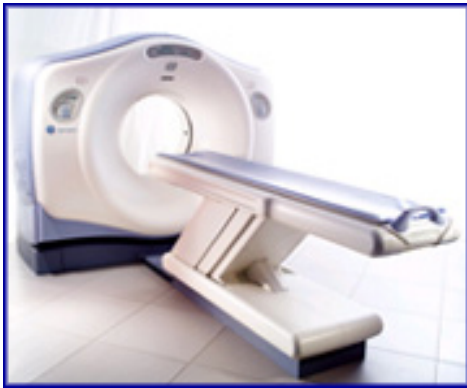
- Anatomical Imaging Modalities
  - Chest X-ray
  - Computed Tomography (CT)
  - Magnetic Resonance Imaging (MRI)
- Functional Imaging Modalities
  - Signal Photon Emission Tomography (SPECT)
  - Positron Emission Tomography (PET)
  - Functional Magnetic Resonance Imaging (fMRI)

## Multislice Helical Scan CT

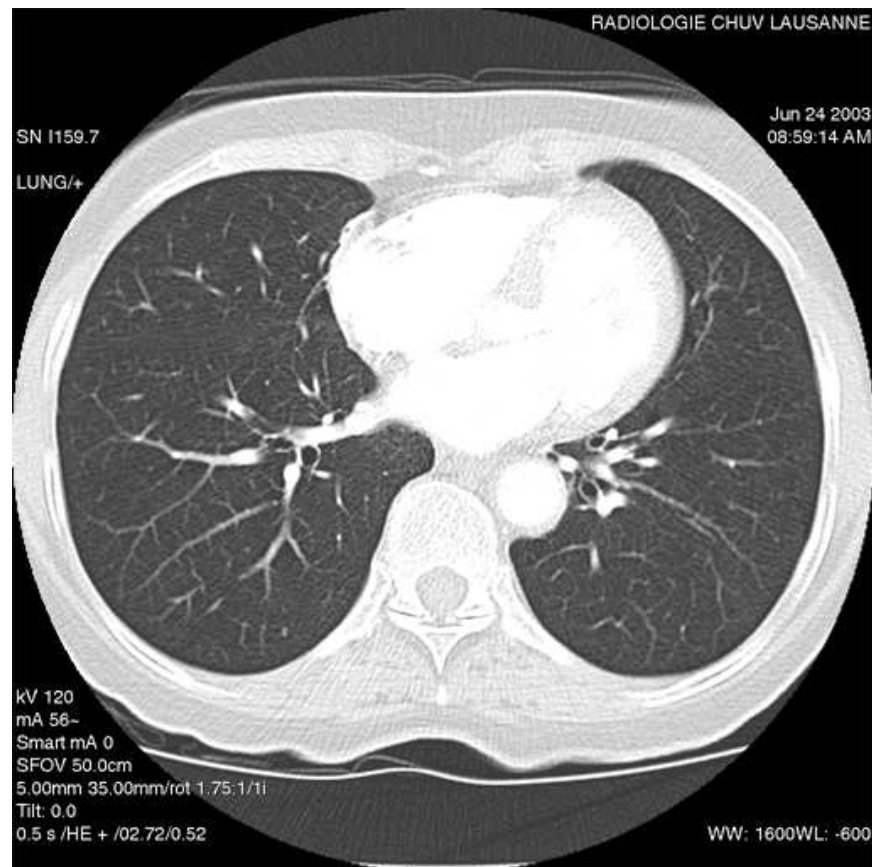
- Multislice CT has a cone-beam structure



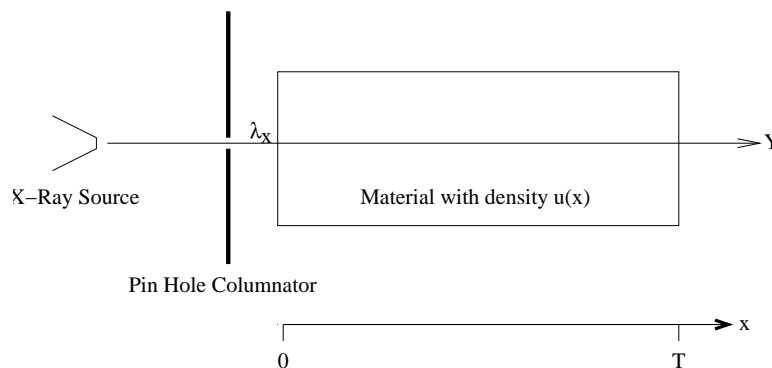
## Example: CT Scan



- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan



## Photon Attenuation



$x$  - depth into material measured in cm

$Y_x$  - Number of photons at depth  $x$

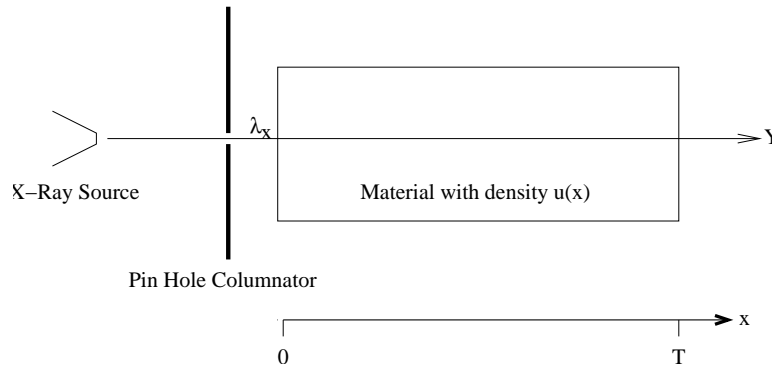
$$\lambda_x = E[Y_x]$$

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x} \lambda_x^k}{k!}.$$

- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.

## Differential Equation for Photon Attenuation



The attenuation of photons obeys the following equation

$$\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x$$

where  $\mu(x)$  is the density in units of  $\text{cm}^{-1}$ .

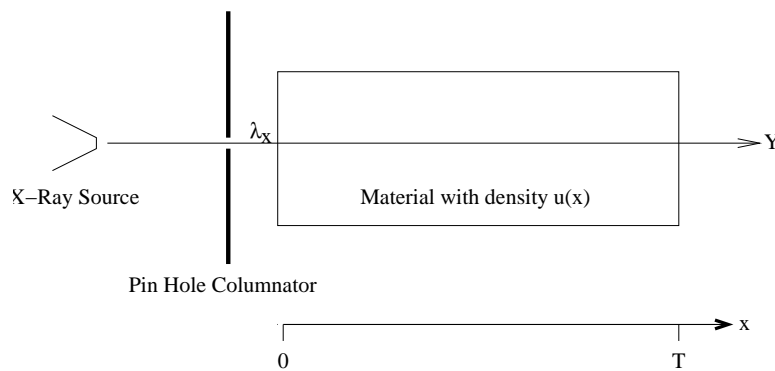
- The solution to this equation is given by

$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t) dt}$$

- So we see that

$$\begin{aligned} \int_0^x \mu(t) dt &= -\log \left( \frac{\lambda_x}{\lambda_0} \right) \\ &\approx -\log \left( \frac{Y_x}{\lambda_0} \right) \end{aligned}$$

## Estimate of the Projection Integral



A commonly used estimate of the projection integral is

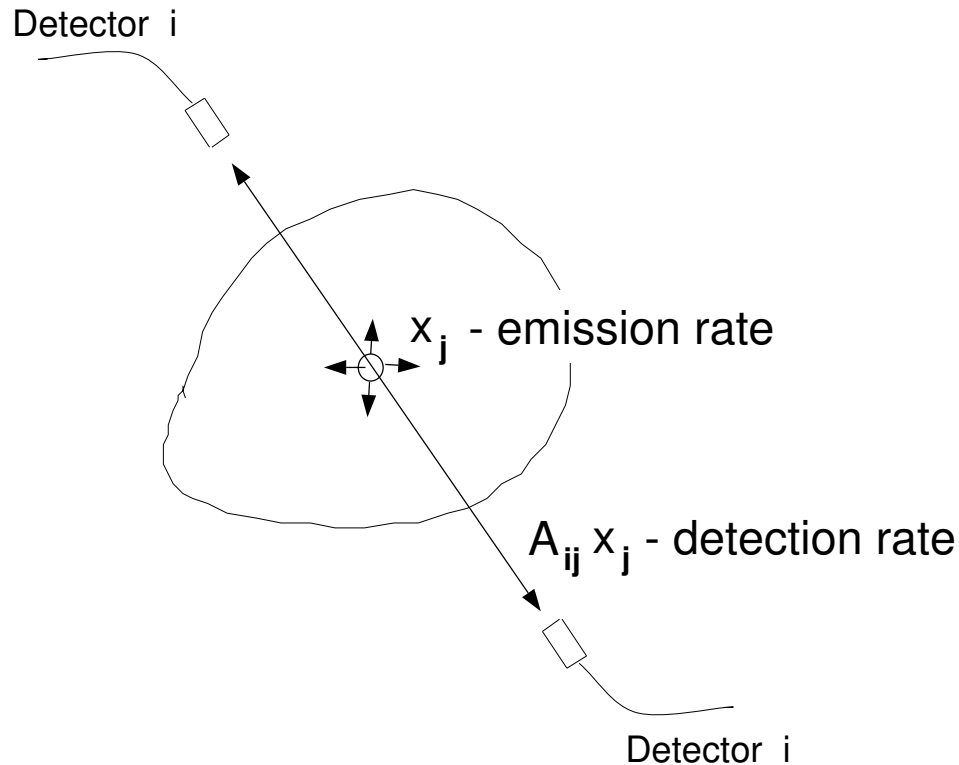
$$\int_0^T \mu(t) dt \approx -\log \left( \frac{Y_T}{\lambda_0} \right)$$

where:

$\lambda_0$  is the dosage

$Y_T$  is the photon count at the detector

## Positron Emission Tomography (PET)

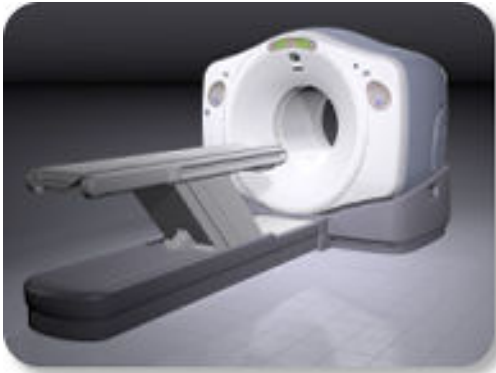


$$E[y_i] = \sum_j A_{ij} x_j$$

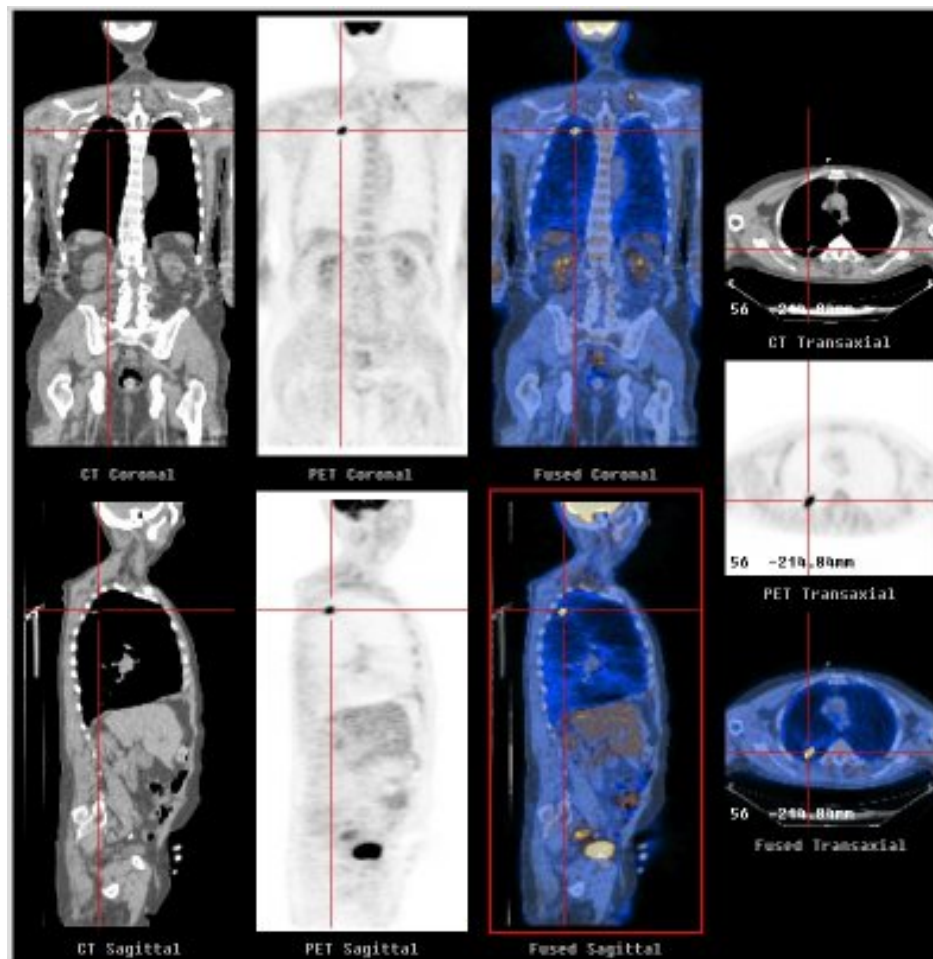
- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements



## Example: PET/CT Scan



- Generally low space/time resolution
- Little anatomical detail  $\Rightarrow$  couple with CT
- Can detect disease



## Coordinate Rotation

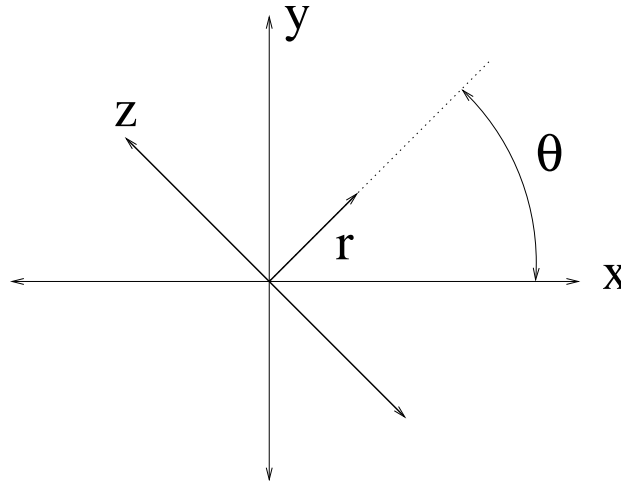
- Define the counter-clockwise rotation matrix

$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Define the new coordinate system  $(r, z)$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$$

- Geometric interpretation

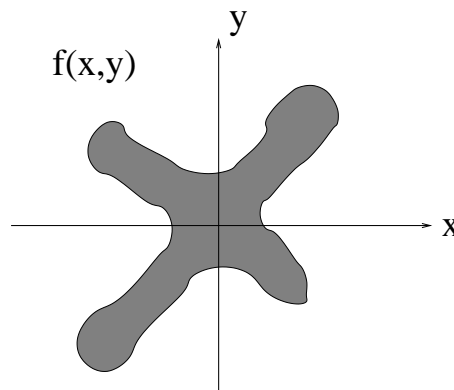


- Inverse transformation

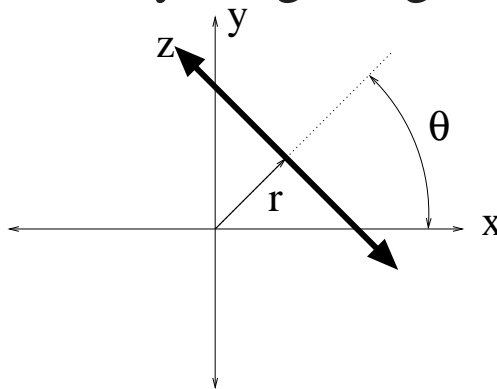
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

## Integration Along Projections

- Consider the function  $f(x, y)$ .



- We compute projections by integrating along  $z$  for each  $r$ .



- The projection integral for each  $r$  and  $\theta$  is given by

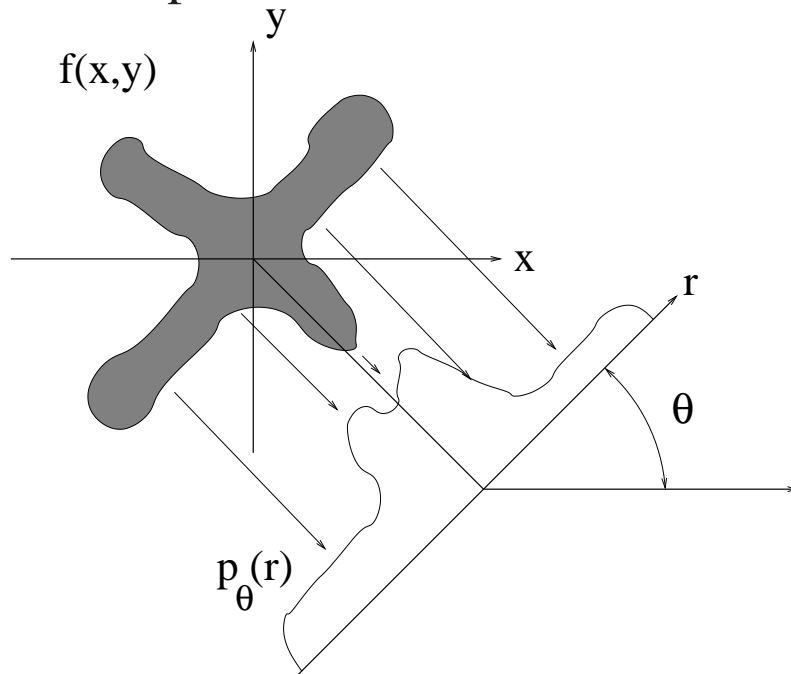
$$\begin{aligned}
 p_{\theta}(r) &= \int_{-\infty}^{\infty} f \left( \mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) dz \\
 &= \int_{-\infty}^{\infty} f (r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta)) dz
 \end{aligned}$$

## The Radon Transform

- The Radon transform of the function  $f(x, y)$  is defined as

$$p_{\theta}(r) = \int_{-\infty}^{\infty} f(r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta)) dz$$

- The geometric interpretation is



Notice that the projection corresponding to  $r = 0$  goes through the point  $(x, y) = (0, 0)$ .

## The Fourier Slice Theorem

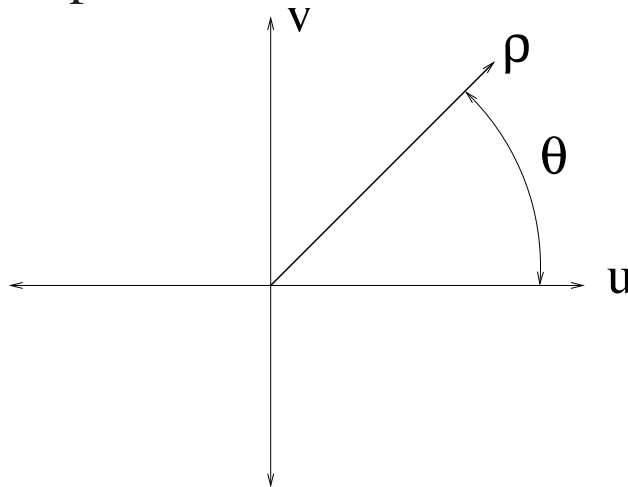
- Let

$$\begin{aligned}P_{\theta}(\rho) &= CTFT \{p_{\theta}(r)\} \\F(u, v) &= CSFT \{f(x, y)\}\end{aligned}$$

Then

$$P_{\theta}(\rho) = F(\rho \cos(\theta), \rho \sin(\theta))$$

- $P_{\theta}(\rho)$  is  $F(u, v)$  in polar coordinates!



## Proof of the Fourier Slice Theorem

- By definition

$$p_{\theta}(r) = \int_{-\infty}^{\infty} f \left( \mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) dz$$

- The CTFT of  $p_{\theta}(r)$  is then given by

$$\begin{aligned} P_{\theta}(\rho) &= \int_{-\infty}^{\infty} p_{\theta}(r) e^{-j2\pi\rho r} dr \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f \left( \mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) dz \right] e^{-j2\pi\rho r} dr \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \left( \mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) e^{-j2\pi\rho r} dz dr \end{aligned}$$

- We next make the change of variables

$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Notice that the Jacobian is  $|\mathbf{A}_{\theta}| = 1$ , and that  $r = x \cos(\theta) + y \sin(\theta)$ . This results in

$$\begin{aligned} P_{\theta}(\rho) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi\rho[x \cos(\theta) + y \sin(\theta)]} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi[x\rho \cos(\theta) + y\rho \sin(\theta)]} dx dy \\ &= F(\rho \cos(\theta), \rho \sin(\theta)) \end{aligned}$$

## Alternative Proof of the Fourier Slice Theorem

- First let  $\theta = 0$ , then

$$p_0(r) = \int_{-\infty}^{\infty} f(r, y) dy$$

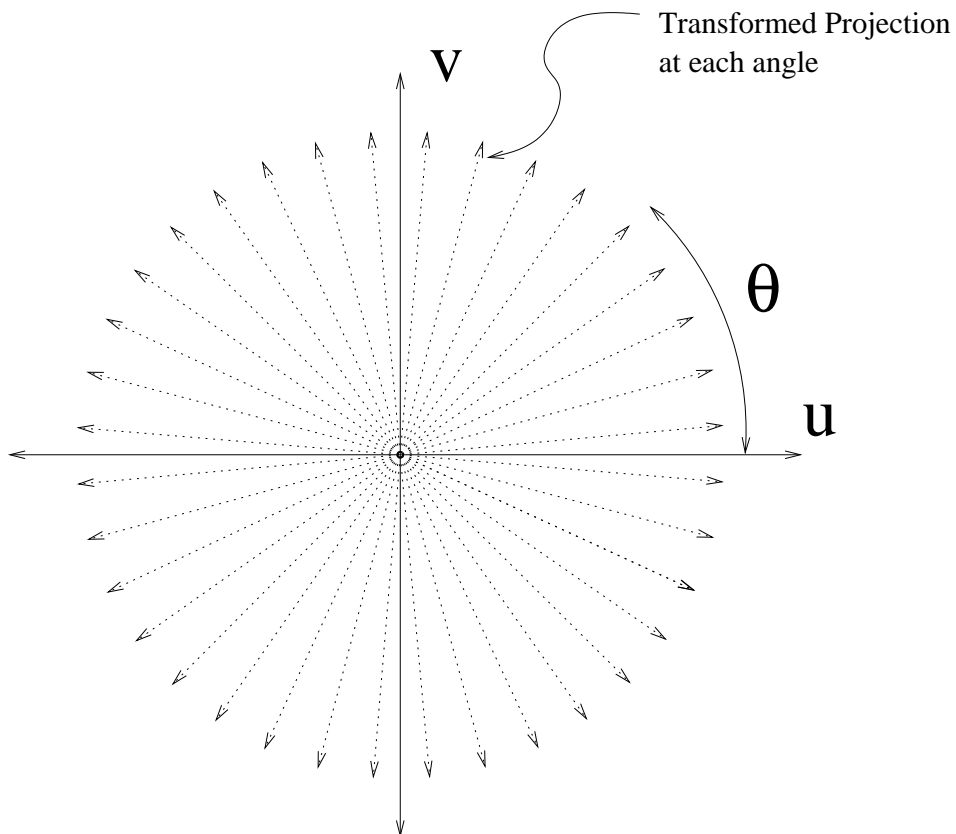
Then

$$\begin{aligned} P_0(\rho) &= \int_{-\infty}^{\infty} p_0(r) e^{-2\pi j r \rho} dr \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(r, y) dy \right] e^{-2\pi j r \rho} dr \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, y) e^{-2\pi j (r\rho + y0)} dr dy \\ &= F(\rho, 0) \end{aligned}$$

- By rotation property of CSFT, it must hold for any  $\theta$ .

## Inverse Radon Transform

- Physical systems measure  $p_\theta(r)$ .
- From these, we compute  $P_\theta(\rho) = CTFT\{p_\theta(r)\}$ .



- Next we take an inverse CSFT to form  $f(x, y)$ .

**Problem:** This requires polar to rectangular conversion.

**Solution:** Convolution backprojection



## Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of  $F(u, v)$  in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du dv = |\rho| d\theta d\rho$$

- This results in the expression

$$\begin{aligned} f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi j(xu+yv)} du dv \\ &= \int_{-\infty}^{\infty} \int_0^{\pi} P_{\theta}(\rho) e^{2\pi j(x\rho \cos(\theta)+y\rho \sin(\theta))} |\rho| d\theta d\rho \\ &= \int_0^{\pi} \left[ \underbrace{\int_{-\infty}^{\infty} |\rho| P_{\theta}(\rho) e^{2\pi j\rho(x \cos(\theta)+y \sin(\theta))} d\rho}_{g_{\theta}(x \cos(\theta)+y \sin(\theta))} \right] d\theta \end{aligned}$$

- Then  $g(t)$  is given by

$$\begin{aligned} g_{\theta}(t) &= \int_{-\infty}^{\infty} |\rho| P_{\theta}(\rho) e^{2\pi j\rho t} d\rho \\ &= CTFT^{-1} \{ |\rho| P_{\theta}(\rho) \} \\ &= h(t) * p_{\theta}(r) \end{aligned}$$

where  $h(t) = CTF T^{-1} \{|\rho|\}$ , and

$$f(x, y) = \int_0^\pi g_\theta (x \cos(\theta) + y \sin(\theta)) d\theta$$

## Summary of CBP Algorithm

1. Measure projections  $p_\theta(r)$ .
2. Filter the projections  $g_\theta(r) = h(r) * p_\theta(r)$ .
3. Back project filtered projections

$$f(x, y) = \int_0^\pi g_\theta (x \cos(\theta) + y \sin(\theta)) d\theta$$

## A Closer Look at Projection Filter

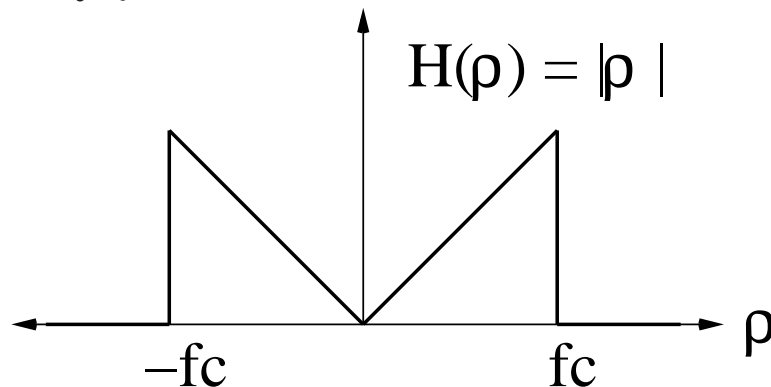
1. At each angle, projections are filtered.

$$g_{\theta}(r) = h(r) * p_{\theta}(r)$$

2. The frequency response of the filter is given by

$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to  $|\rho| \leq f_c$  for some cut-off frequency  $f_c$ .



So

$$H(\rho) = f_c [\text{rect}(f/(2f_c)) - \Lambda(f/f_c)]$$

$$h(r) = f_c^2 [2\text{sinc}(t2f_c) - \text{sinc}^2(tf_c)]$$

## A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^\pi b_\theta(x, y) d\theta$$

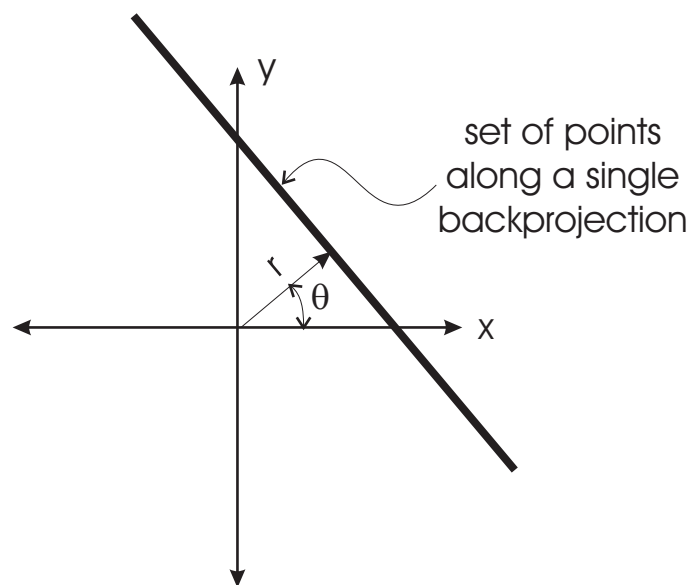
where

$$b_\theta(x, y) = g_\theta(x \cos(\theta) + y \sin(\theta))$$

- Consider the set of points  $(x, y)$  such that

$$r = x \cos(\theta) + y \sin(\theta)$$

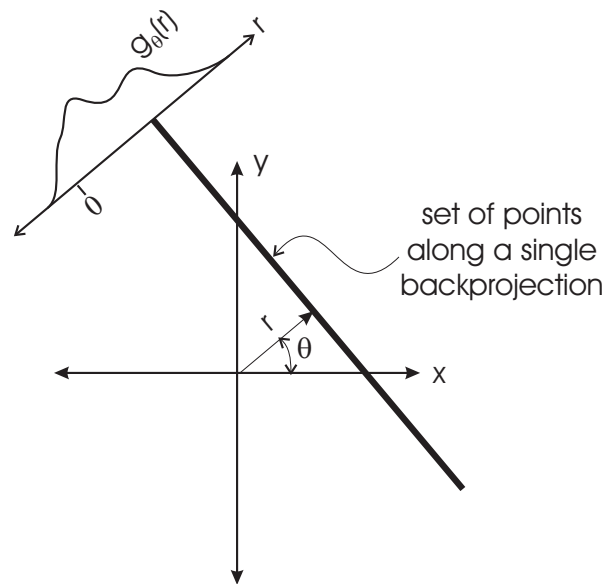
This set looks like



- Along this line  $b_\theta(x, y) = g_\theta(r)$ .

## Back Projection Continued

- For each angle  $\theta$  back projection is constant along lines of angle  $\theta$  and takes on value  $g_\theta(r)$ .



- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to  $\pi$ .

$$f(x, y) = \int_0^\pi b_\theta(x, y) d\theta$$

$$\approx \frac{\pi}{M} \sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x, y)$$

- Back projection “smears” values of  $g(r)$  back over image, and then adds smeared images for each angle.