PURDUE

ECE 63700

Final Exam, May 1, Spring 2023

NAME ____________________________________

PUID ____

Exam instructions:

- A fact sheet is included at the end of this exam for your use.
- You have 120 minutes to work the exam.
- This is a closed-book and closed-note exam. You may not use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You may not communicate with any person other than the official proctor during the exam.

To ensure Gradescope can read your exam:

- Write your full name and PUID above and on the top of every page.
- Answer all questions in the area designated for each problem.
- Write only on the front of the exam pages.
- DO NOT run over to the next question.

Name/PUID: Key

Problem 1. (30pt) Least Squares Linear Filtering

Let x_s be the value of a pixel in an image where s is a point in a 2D rectangular lattice, and

$$z_s = \{x_{s-r_0}, \cdots, x_{s-r_{p-1}}\}$$

is a array of p pixels that surround s but do not include s.

Then define the full rank matrix

$$Z = \begin{bmatrix} z_{s_0} \\ \vdots \\ z_{s_N} \end{bmatrix}$$

and the column vector

$$X = \begin{bmatrix} x_{s_0} \\ \vdots \\ x_{s_N} \end{bmatrix}$$

where $S = \{s_0, \cdots, s_{N-1}\}$ is a set of N pixels selected from a training image. Further more, let $\hat{x}_s = z_s\theta$ be an estimate of x_s where θ is a parameter vector of dimension p.

Problem 1a) Calculate an expression for a loss function, $L(\theta)$, given by the total squared error between x_s and $\hat{x}_s$ on S.

Solution:

$$L(\theta) = \frac{1}{|S|}\sum_{s\in S}(x_s - \hat{x}_s)^2 = \frac{1}{|S|}\|X - Z\theta\|^2$$

Problem 1b) Calculate a closed form expression the value θ^* that minimizes the loss function $L(\theta)$.

Solution:

$$\begin{aligned} 0 &= \nabla_\theta L(\theta) \\ &= \frac{1}{|S|}\nabla_\theta \|X - Z\theta\|^2 \\ &= \frac{1}{|S|}\nabla_\theta (X - Z\theta)^t(X - Z\theta) \\ &= -\frac{1}{|S|}(X - Z\theta)^t Z \end{aligned}$$

So from this, we have that

$$\begin{aligned} 0 &= -\frac{1}{|S|}Z^t(X - Z\theta) \\ &= -\frac{1}{|S|}Z^t X - Z^t Z\theta \end{aligned}$$

So then

$$\frac{1}{|S|}Z^t X = \frac{1}{|S|}Z^t Z\theta$$

Which yields the result that

$$\theta^* = \hat{R}^{-1}\hat{b}$$

where

$$\begin{aligned} \hat{R} &= \frac{1}{|S|}Z^t Z \\ \hat{b} &= \frac{1}{|S|}Z^t X \end{aligned}$$

Problem 1c) Calculate a closed form expression the value $\hat{X}$ that minimizes the squared error between $\hat{X}$ and X.

Solution:

$$\hat{X} = Z\theta^* = Z\hat{R}^{-1}\hat{b} \ ,$$

where

$$\begin{aligned} \hat{R} &= \frac{1}{|S|}Z^tZ \\ \hat{b} &= \frac{1}{|S|}Z^tX \end{aligned}$$

Problem 1d) Calculate a closed form expression the value $L(\theta^*)$.

Solution:

$$\begin{aligned} L(\theta^*) &= \frac{1}{|S|}\|X - Z\theta^*\|^2 \\ &= \frac{1}{|S|}\|X - Z\hat{R}^{-1}\hat{b}\|^2 \\ &= \frac{1}{|S|}\|X - Z\left(Z^tZ\right)^{-1}Z^tX\|^2 \end{aligned}$$

where $\hat{R} = \frac{1}{|S|}Z^tZ$ and $\hat{b} = \frac{1}{|S|}Z^tX$.

Optional extended solution: If we define,

$$P = Z\left(Z^tZ\right)^{-1}Z^t \ ,$$

then P symmetric and positive definite projection matrix with $N-p$ eigenvalues of 0 and p eigenvalues of 1 that projects X onto the column space of Z. So then

$$L(\theta^*) = \frac{1}{|S|}\left\{\|X\|^2 - \|PX\|^2\right\} \ .$$

Problem 1e) Describe in words the difference between the Least Squares estimate of θ and the Minimum Mean Squared Error estimate of θ.

Solution: The Least Squares estimate determines the value of θ determines the value that minimizes the squared error on the actual sample values. The Minimum Mean Squared estimate minimizes the expected value of the squared error.

Problem 1f) What is the advantage of the Least Squares estimate over the Minimum Mean Squared Error estimate.

Solution: The Least Squares estimate does not require that you actually know the distribution of the random variables since you can compute it using the observed sample values.

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Problem 2. (35pt) Electro-Magnetic Plane Waves

Consider the following partial differential equation (PDE)

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \ ,$$

that describes the propagation of electromagnetic fields in free space.

Problem 2a) Show that the following function is a solution to this PDE,

$$E_t(x, y, z, t) = E_o \exp\{j2\pi(v \cdot r)\} \exp\{-j2\pi f_o t\} \ .$$

where $v = (v_x, v_y, v_z)$ and $r = (x, y, z)$.

Solution: Notice that

$$\begin{aligned}
\nabla^2 E(x,y,z,t) &= \frac{\partial^2 E(x,y,z,t)}{\partial x^2} + \frac{\partial^2 E(x,y,z,t)}{\partial y^2} + \frac{\partial^2 E(x,y,z,t)}{\partial z^2} \\
&= (j2\pi)^2(v_x^2 + v_y^2 + v_z^2) E_o \exp\{j2\pi(v \cdot r)\} \exp\{-j2\pi f_o t\} \\
&= -(2\pi\|v\|)^2 E_o \exp\{j2\pi(v \cdot r)\} \exp\{-j2\pi f_o t\} \\
&= -(2\pi\|v\|)^2 E(x,y,z,t) \ .
\end{aligned}$$

Also that

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 E(x,y,z,t)}{\partial t^2} &= \frac{1}{c^2}(-j2\pi f_o)^2 E(x,y,z,t) \\
&= -\frac{1}{c^2}(2\pi f_o)^2 E(x,y,z,t)
\end{aligned}$$

So then, the PDE is solved if

$$0 = \left[-(2\pi\|v\|)^2 + \frac{1}{c^2}(2\pi f_o)^2\right] E(x,y,z,t) \ .$$

So this means that

$$\frac{1}{c^2}(2\pi f_o)^2 = (2\pi\|v\|)^2 \ ,$$

which is true if and only if

$$c = \frac{f_o}{\|v\|} \ .$$

Problem 2b) For the solution of part 2a), what constraints must hold on v and f_o?

Solution: We must have that

$$\frac{f_o}{\|v\|} = c \ .$$

Problem 2c) For an electro-magnetic wave in free space, what is the interpretation of the constant c?

Solution: c is the speed of light.

Problem 2d) How far does light travel in free space in 1 nano second?

Solution: Approximately 1 ft or 30 cm.

Problem 2e) If $f_o = 10GHz$, then what is the plane-wave's wavelength in free space?

Solution: Approximately 3 cm or 1.2 in.

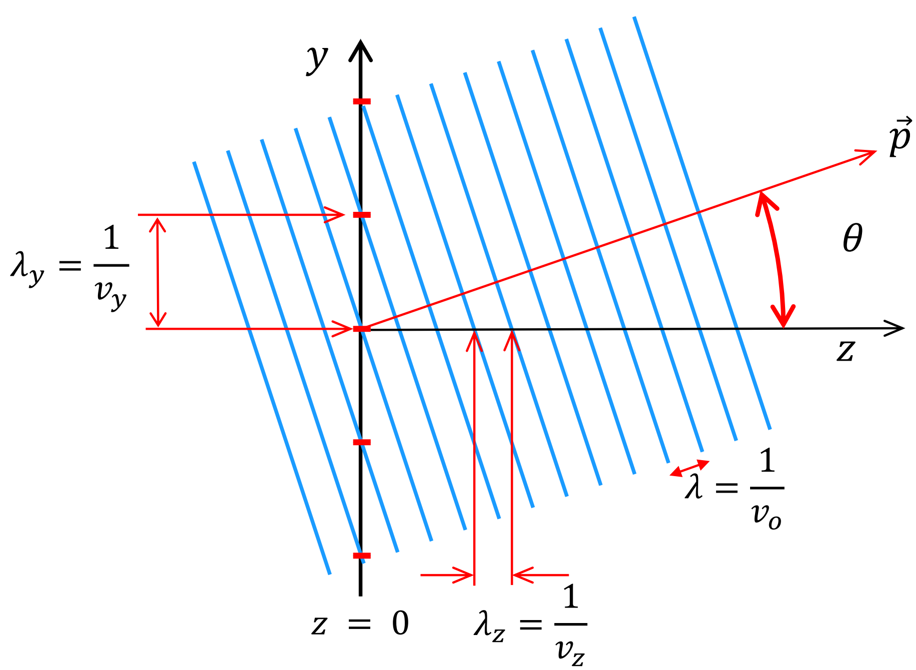
Problem 2f) If $E_o = A_o e^{j\theta}$, then derive an expression for the real value of the electric field of the plane wave as a function of r and t.

Solution:

$$
\begin{aligned}
\text{Electric Field} &= Re\{E_t(x,y,z,t)\} \\
&= Re\{E_o \exp\{j2\pi(v\cdot r)\}\exp\{-j2\pi f_o t\}\} \\
&= Re\{A_o e^{j\theta}\exp\{j2\pi(v\cdot r)\}\exp\{-j2\pi f_o t\}\} \\
&= Re\{A_o \exp\{j2\pi(v\cdot r) - j2\pi f_o t + \theta\}\} \\
&= A_o Re\{\exp\{j2\pi(v\cdot r - f_o t) + \theta\}\} \\
&= A_o \cos\{2\pi(v\cdot r - f_o t) + \theta\}
\end{aligned}
$$

Problem 2g) Sketch the plane wave in (y,z) when $v_x = 0$ for $v_y = v_o\sin(\theta)$ and $v_z = v_o\cos(\theta)$.

Solution:



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Problem 3. (35pt) Discrete-Time Low Pass Filter

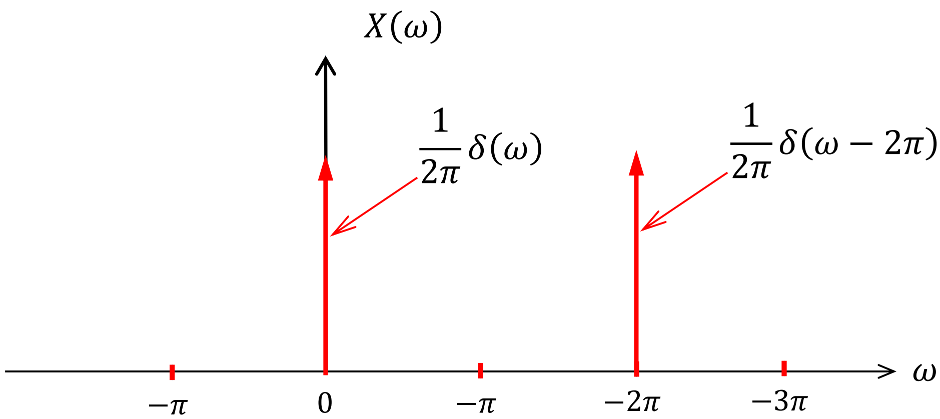
Consider a discrete time filter given by

$$y_n = \sum_{k=-\infty}^{\infty} h_{n-k} x_k \ .$$

Furthermore, let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ be the DTFT of x_n, y_n, and h_n, respectively.

Problem 3a) Assuming that $X(\omega) = \frac{1}{2\pi}\delta(\omega)$ for $|\omega| \leq \pi$, then sketch $X(\omega)$ for $-\pi \leq \omega \leq 3\pi$.

Solution:



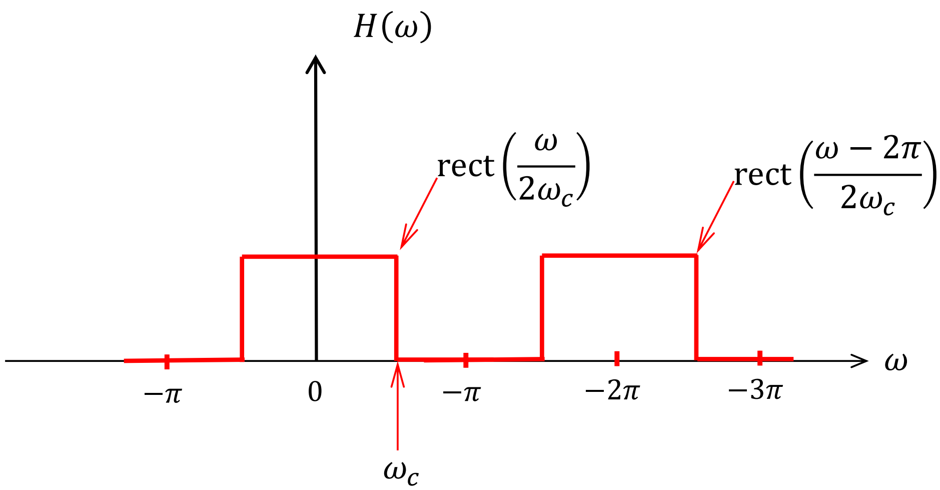
Problem 3b) Assuming that $X(\omega) = 2\pi\delta(\omega)$ for $|\omega| \leq \pi$, then calculate x_n.

Solution:

$$x_n = 1$$

Problem 3c) For this part, assume that this is a low pass filter with cut-off frequency ω_c, unity gain in the band pass, and $0 < \omega_c < \pi$. Then sketch the function $H(\omega)$ for $-\pi \leq \omega \leq 3\pi$.

Solution:



Problem 3d) For this part, assume that this is a low pass filter with cut-off frequency ω_c, unity gain in the band pass, and $0 < \omega_c < \pi$. Then calculate the function h_n.

Solution:

$$h_n = T\text{sinc}(Tn) \ ,$$

where $T = \frac{\omega_c}{\pi}$.

Problem 3e) Assuming that $\omega_c = \pi/2$, then calculate h_n and sketch the function.

Solution:

$$h_n = \text{sinc}(n/2) \ .$$

Problem 3f) In a streaming application, the signal x_n comes into your system in a continuous stream that is effectively never ending. In this case, what must you do to implement the discrete time filter? Why?

Solution: You must window it to a finite window otherwise it would require an infinite amount of computation to compute each output point.

Problem 3g) If you must window the signal h_n, than what is the **worse possible** window to use? Why?

Solution: A rectangular window is the worse because it creates large side-lobes in the frequency domain.

Name/PUID: ______________________________

Problem 4. (25pt) Tomography

Consider the 2D system given by

$$g(x, y) = f(x, y) \circ h(x, y)$$

where $\circ$ represents 2D convolution and

$$h(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \ .$$

A common problem in image processing is called "deconvolution". The objective of deconvolution is to recover the signal $f(x, y)$ from knowledge of $g(x, y)$ and $h(x, y)$.

Problem 4a) Use the Fourier transform to show that

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi rt} dr \ .$$

Solution: If we take $x(t) = \delta(t)$, then we have that

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft} dt = 1 \ .$$

Therefore, we know that

$$x(t) = \delta(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} 1\, e^{j2\pi ft} df \ .$$

So then we have that

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} df \ .$$

Problem 4b) Use the result of a) to show that

$$\delta(u\cos\theta + v\sin\theta) = \int_{-\infty}^{\infty} e^{j2\pi r(u\cos\theta + v\sin\theta)} dr \ .$$

Solution: Substituting in $t = u\cos\theta + v\sin\theta$, we get that

$$\delta(u\cos\theta + v\sin\theta) = \int_{-\infty}^{\infty} e^{j2\pi f(u\cos\theta + v\sin\theta)} df \ .$$

Problem 4c) Show that the Fourier transform of $h(x, y)$ has the form

$$H(u, v) = \int_{-\infty}^{\infty}\int_{-\pi/2}^{\pi/2} \frac{1}{r} e^{j2\pi(ur\cos\theta + vr\sin\theta)} r d\theta dr = \int_{-\pi/2}^{\pi/2} \delta(u\cos\theta + v\sin\theta) d\theta$$

Solution:

$$\begin{aligned} H(u, v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y) \exp\left\{-j2\pi(ux + vy)\right\} dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + y^2}} \exp\left\{-j2\pi(ux + vy)\right\} dx\, dy \ . \end{aligned}$$

Then we can make the change of variables $(x, y) = (r\cos\theta, r\sin\theta)$. This results in

$$\begin{aligned} H(u, v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + y^2}} \exp\left\{-j2\pi(ux + vy)\right\} dx\, dy \\ &= \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty} \frac{1}{r} \exp\left\{-j2\pi(ur\cos\theta + vr\sin\theta)\right\} r dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty} \exp\left\{-j2\pi r(u\cos\theta + v\sin\theta)\right\} dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_{-\infty}^{\infty} \exp\left\{-j2\pi r(u\cos\theta + v\sin\theta)\right\} dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \delta(u\cos\theta + v\sin\theta)\, d\theta \ . \end{aligned}$$

Problem 4d) Use this result to show that

$$H(u,v) = \frac{1}{\sqrt{u^2+v^2}} \ .$$

Solution: Define $\omega = (u,v)$, $\rho = \|\omega\|$, and $\phi = \arctan(u,v)$ so that $f = (\rho\cos\phi, \rho\sin\phi)$. In other words, (ρ,ϕ) is a polar coordinate representation of $f = (u,v)$.

Then

$$\begin{aligned}
H(u,v) = H(f) &= \int_{-\pi/2}^{\pi/2} \delta(\omega\cdot(\cos\theta, v\sin\theta))\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \delta(\rho\cos(\theta-\phi))\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \delta(\rho\sin(\theta-\phi-\pi/2))\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \delta(\rho\sin(\theta-\phi-\pi/2))\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \delta(\rho\sin(\theta-\phi'))\, d\theta \ .
\end{aligned}$$

where $\phi' = \phi - \pi/2$.

There are two cases to consider here. Case 1: $-\pi/2 < \phi' < \pi/2$ and Case 2: $\pi/2 < \phi' < 3\pi/2$. In Case 1,

$$\begin{aligned}
H(u,v) = H(f) &= \int_{-\pi/2}^{\pi/2} \delta(\rho\sin(\theta-\phi'))\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \delta(\rho(\theta-\phi'))\, d\theta \\
&= \int_{-\infty}^{\infty} \delta(\rho(\theta-\phi'))\, d\theta \\
&= \int_{-\infty}^{\infty} \delta(\rho\,\theta)\, d\theta \\
&= \int_{-\infty}^{\infty} \delta(x)\, \frac{1}{\rho} dx \\
&= \frac{1}{\rho}\int_{-\infty}^{\infty} \delta(x) dx = \frac{1}{\rho} = \frac{1}{\sqrt{u^2+v^2}} \ .
\end{aligned}$$

Case 2 is similar.

Problem 4e) Which task is more difficult: Deconvolution with $h(x, y) = \frac{1}{\sqrt{x^2+y^2}}$; or deconvolution with $h(x, y) = \mathrm{sinc}(x, y)$? Justify your answer.

Solution: Deconvolution with $\mathrm{sinc}(x, y)$ is more difficult. Both functions drop off as $\frac{1}{\|f\|}$ in the frequency domain, but the sinc function has nulls that in frequency that are impossible to recover.

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Problem 5. (25pt) Sampling

Let $g(t) = \text{sinc}(t)$ and let $s(n) = g(Tn)$ be a sampled version of $g(t)$ where $T > 0$.

Problem 5a) Calculate $G(f)$ the CTFT of $g(t)$.

Solution:

$$G(f) = \text{rect}(f)$$

Problem 5b) Write an expression for $S(e^{j\omega})$ the DTFT of $s(n)$.

Solution:

$$\begin{aligned} S(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} G\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 2\pi k}{2\pi T}\right) \end{aligned}$$

Problem 5c) What is the constraint on T that ensures that $g(t)$ can be perfectly reconstructed.

Solution:

$$T < 1$$

Problem 5d) Calculate and sketch $s(n)$ and $S(e^{j\omega})$ for $T = 1$.

Solution:

$$\begin{aligned} s(n) = &= \text{sinc}(n) = \delta(n) \\ S(e^{j\omega}) &= 1 \text{ for } |\omega| < \pi \end{aligned}$$

Problem 5e) Calculate and sketch $s(n)$ and $S(e^{j\omega})$ for $T = 1/2$.

Solution:

$$\begin{aligned} s(n) &= \text{sinc}(n/2) = \delta(n) \\ S(e^{j\omega}) &= \text{rect}(\omega/\pi) \text{ for } |\omega| < \pi \end{aligned}$$

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\mathrm{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$