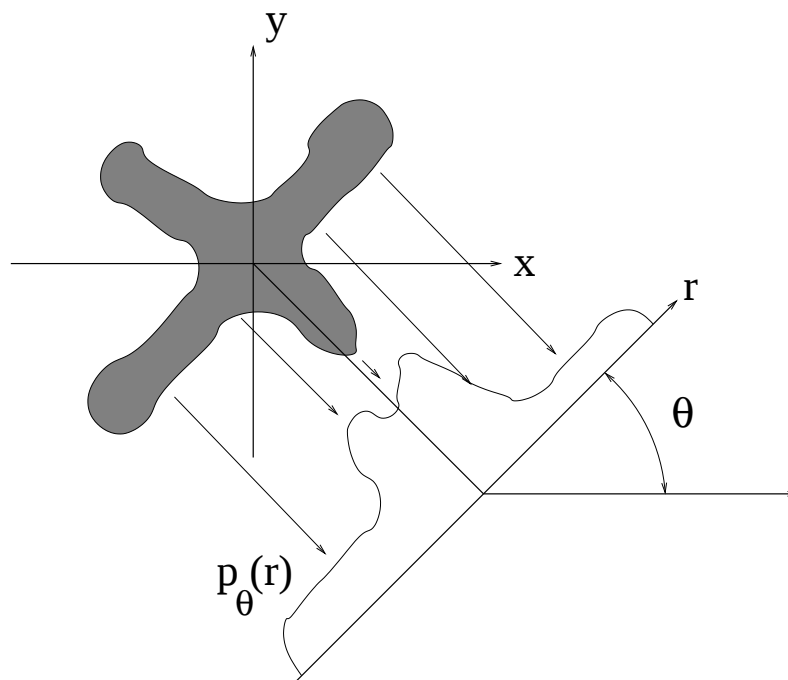


Tomography

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.
 - Projections must be collected at every angle θ and displacement r .
 - Forward projections $p_\theta(r)$ are known as a Radon transform.



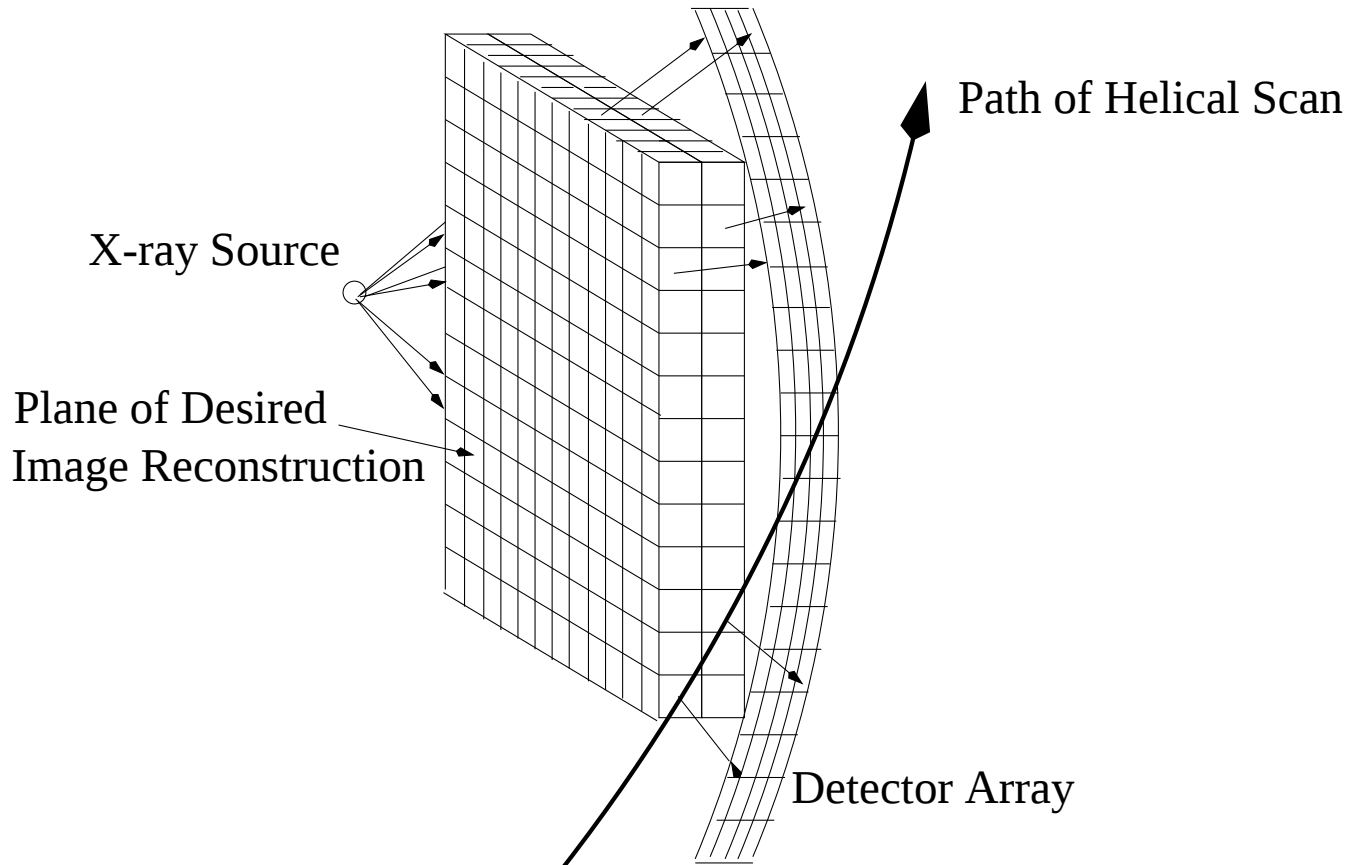
- Objective: reverse this process to form the original image $f(x, y)$.
 - Fourier Slice Theorem is the basis of inverse
 - Inverse can be computed using convolution back projection (CBP)

Medical Imaging Modalities

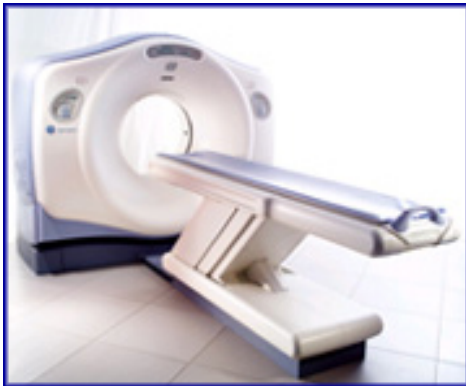
- Anatomical Imaging Modalities
 - Chest X-ray
 - Computed Tomography (CT)
 - Magnetic Resonance Imaging (MRI)
- Functional Imaging Modalities
 - Signal Photon Emission Tomography (SPECT)
 - Positron Emission Tomography (PET)
 - Functional Magnetic Resonance Imaging (fMRI)

Multislice Helical Scan CT

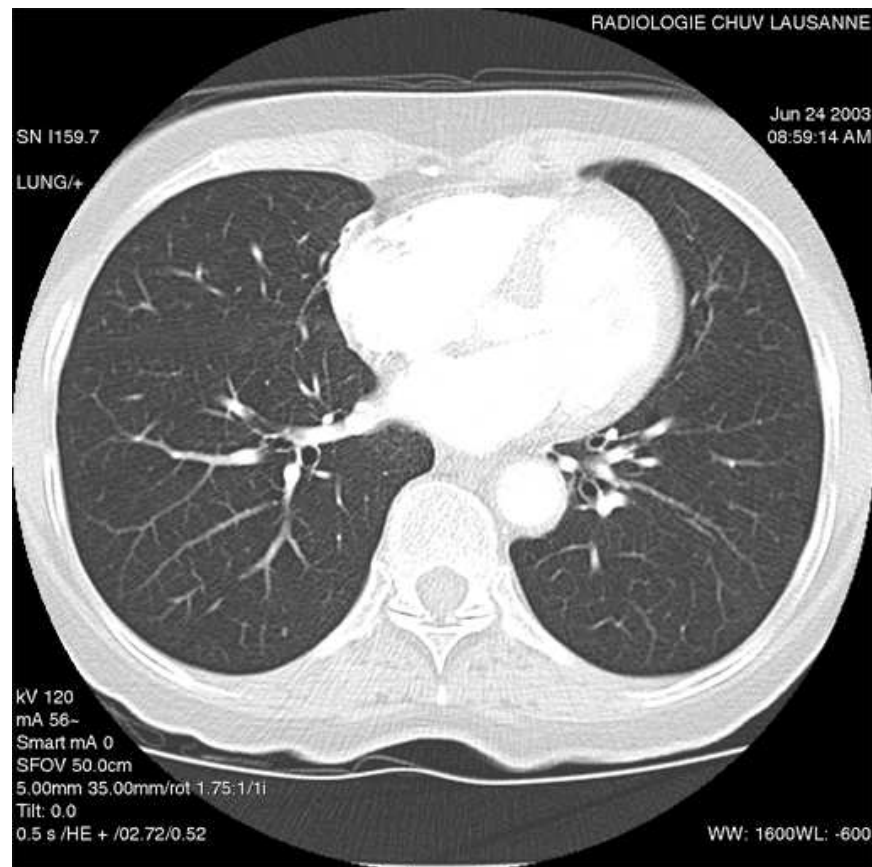
- Multislice CT has a cone-beam structure



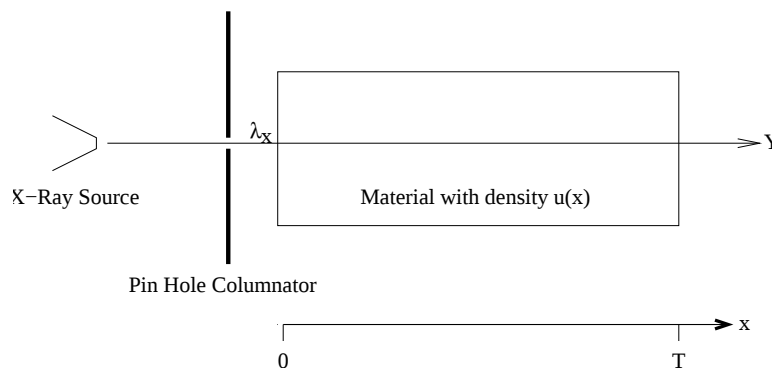
Example: CT Scan



- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan



Photon Attenuation



x - depth into material measured in cm

Y_x - Number of photons at depth x

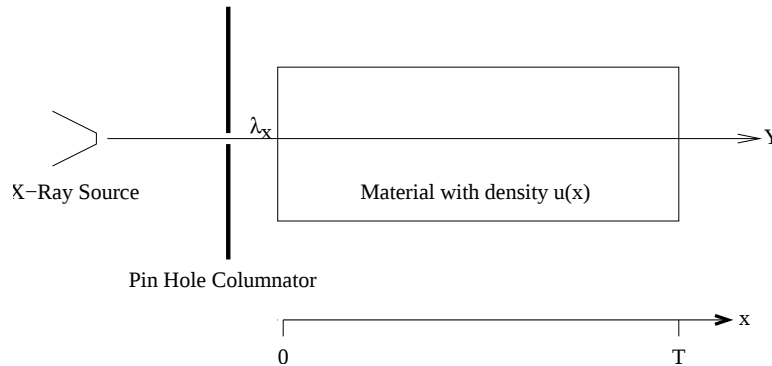
$$\lambda_x = E[Y_x]$$

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x} \lambda_x^k}{k!}.$$

- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.

Differential Equation for Photon Attenuation



The attenuation of photons obeys the following equation

$$\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x$$

where $\mu(x)$ is the density in units of cm^{-1} .

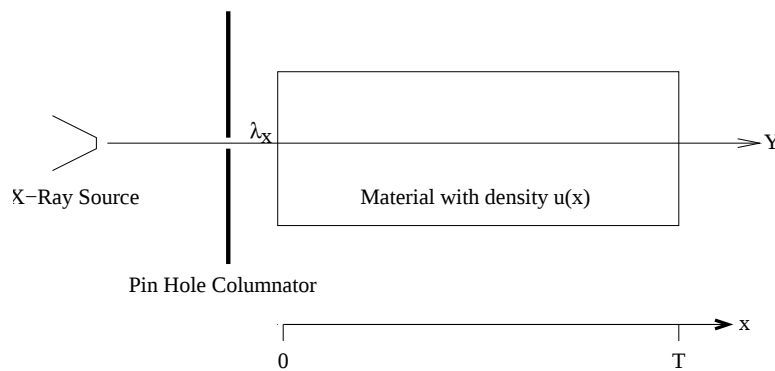
- The solution to this equation is given by

$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t) dt}$$

- So we see that

$$\begin{aligned} \int_0^x \mu(t) dt &= -\log \left(\frac{\lambda_x}{\lambda_0} \right) \\ &\approx -\log \left(\frac{Y_x}{\lambda_0} \right) \end{aligned}$$

Estimate of the Projection Integral



A commonly used estimate of the projection integral is

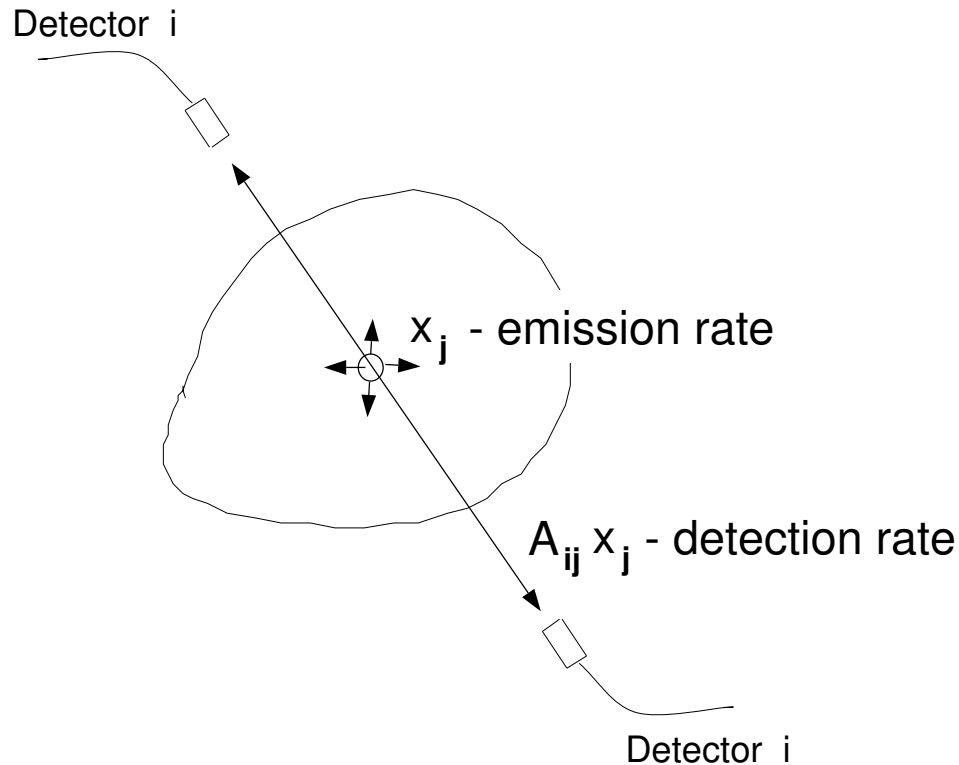
$$\int_0^T \mu(t) dt \approx -\log \left(\frac{Y_T}{\lambda_0} \right)$$

where:

λ_0 is the dosage

Y_T is the photon count at the detector

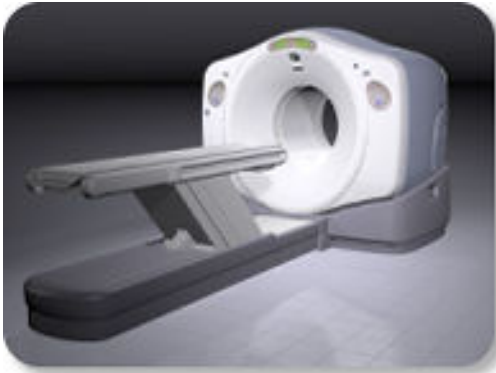
Positron Emission Tomography (PET)



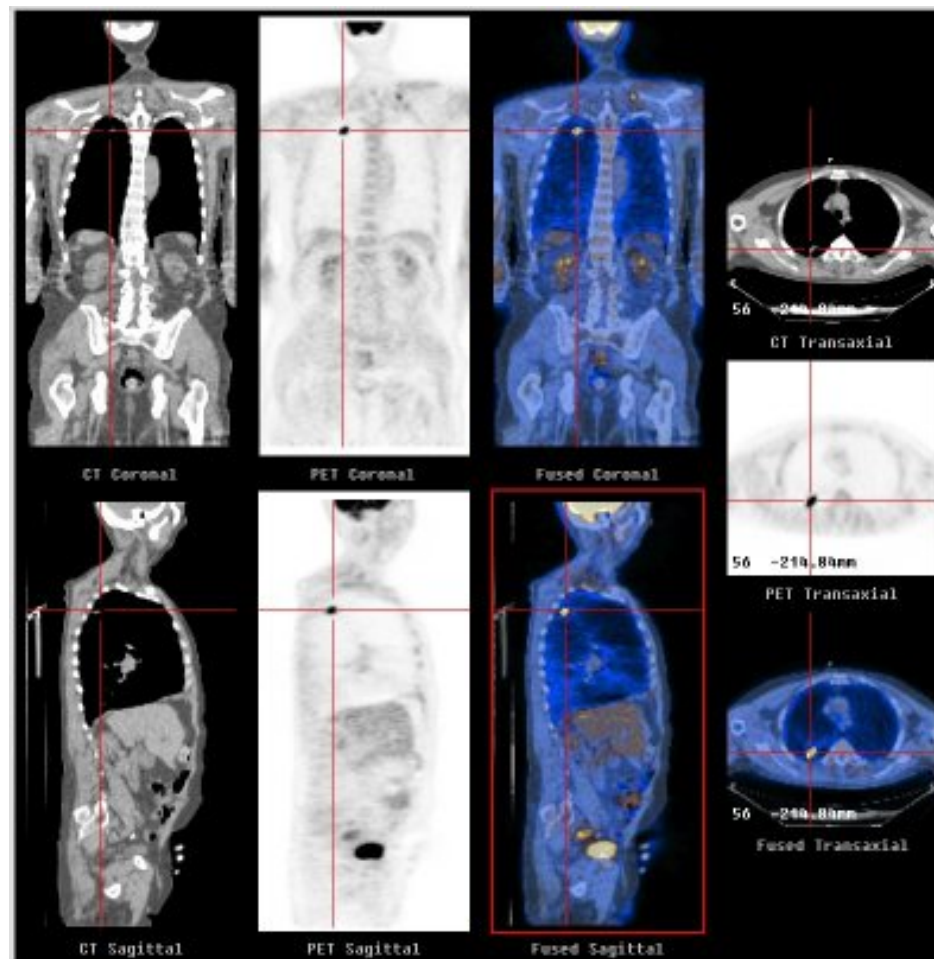
$$E[y_i] = \sum_j A_{ij} x_j$$

- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements

Example: PET/CT Scan



- Generally low space/time resolution
- Little anatomical detail $\Rightarrow$ couple with CT
- Can detect disease



Coordinate Rotation

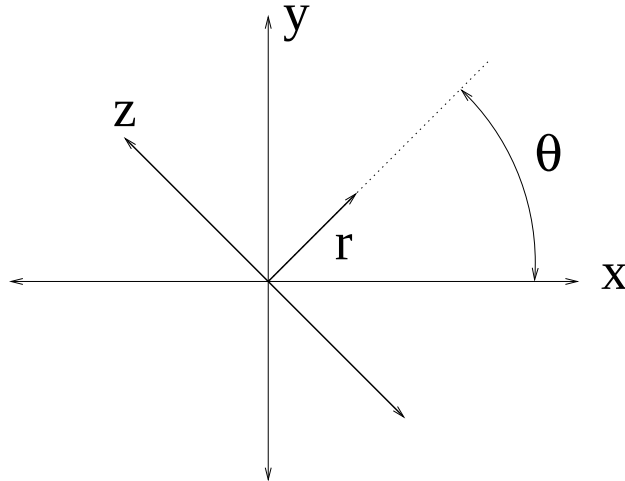
- Define the counter-clockwise rotation matrix

$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Define the new coordinate system (r, z)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$$

- Geometric interpretation

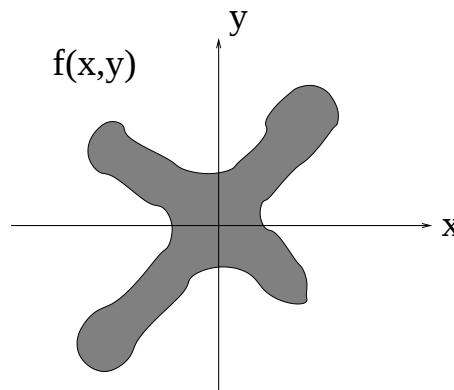


- Inverse transformation

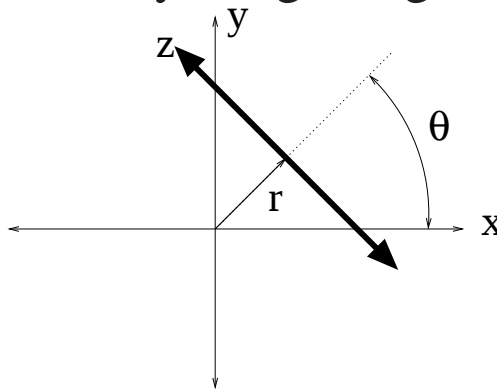
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

Integration Along Projections

- Consider the function $f(x, y)$.



- We compute projections by integrating along z for each r .



- The projection integral for each r and θ is given by

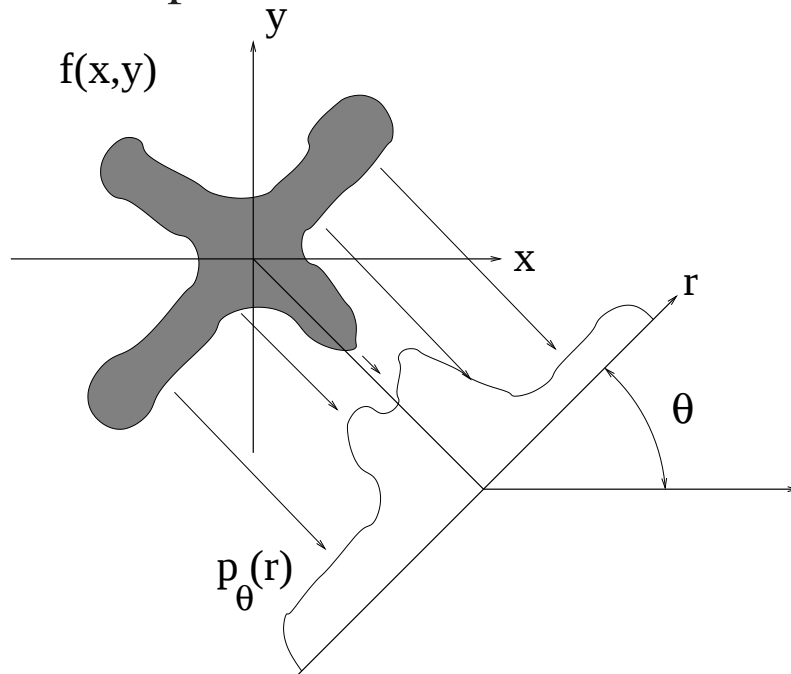
$$\begin{aligned}
 p_{\theta}(r) &= \int_{-\infty}^{\infty} f \left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) dz \\
 &= \int_{-\infty}^{\infty} f (r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta)) dz
 \end{aligned}$$

The Radon Transform

- The Radon transform of the function $f(x, y)$ is defined as

$$p_{\theta}(r) = \int_{-\infty}^{\infty} f(r \cos(\theta) - z \sin(\theta), r \sin(\theta) + z \cos(\theta)) dz$$

- The geometric interpretation is



Notice that the projection corresponding to $r = 0$ goes through the point $(x, y) = (0, 0)$.

The Fourier Slice Theorem

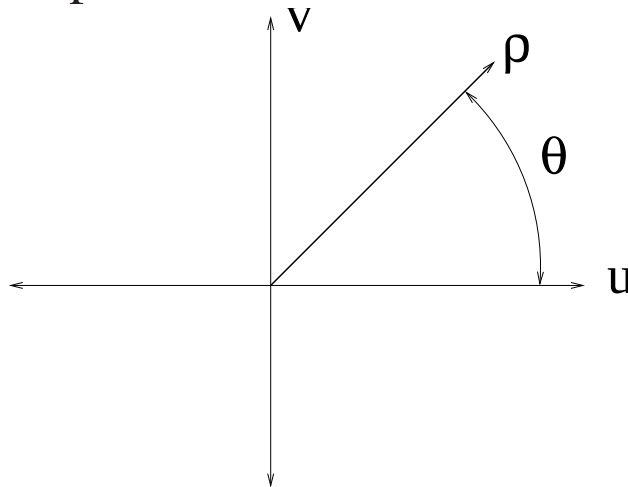
- Let

$$\begin{aligned}P_{\theta}(\rho) &= CTFT \{p_{\theta}(r)\} \\F(u, v) &= CSFT \{f(x, y)\}\end{aligned}$$

Then

$$P_{\theta}(\rho) = F(\rho \cos(\theta), \rho \sin(\theta))$$

- $P_{\theta}(\rho)$ is $F(u, v)$ in polar coordinates!



Proof of the Fourier Slice Theorem

- By definition

$$p_{\theta}(r) = \int_{-\infty}^{\infty} f \left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) dz$$

- The CTFT of $p_{\theta}(r)$ is then given by

$$\begin{aligned} P_{\theta}(\rho) &= \int_{-\infty}^{\infty} p_{\theta}(r) e^{-j2\pi\rho r} dr \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f \left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) dz \right] e^{-j2\pi\rho r} dr \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix} \right) e^{-j2\pi\rho r} dz dr \end{aligned}$$

- We next make the change of variables

$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Notice that the Jacobian is $|\mathbf{A}_{\theta}| = 1$, and that $r = x \cos(\theta) + y \sin(\theta)$. This results in

$$\begin{aligned} P_{\theta}(\rho) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi\rho[x \cos(\theta) + y \sin(\theta)]} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi[x\rho \cos(\theta) + y\rho \sin(\theta)]} dx dy \\ &= F(\rho \cos(\theta), \rho \sin(\theta)) \end{aligned}$$

Alternative Proof of the Fourier Slice Theorem

- First let $\theta = 0$, then

$$p_0(r) = \int_{-\infty}^{\infty} f(r, y) dy$$

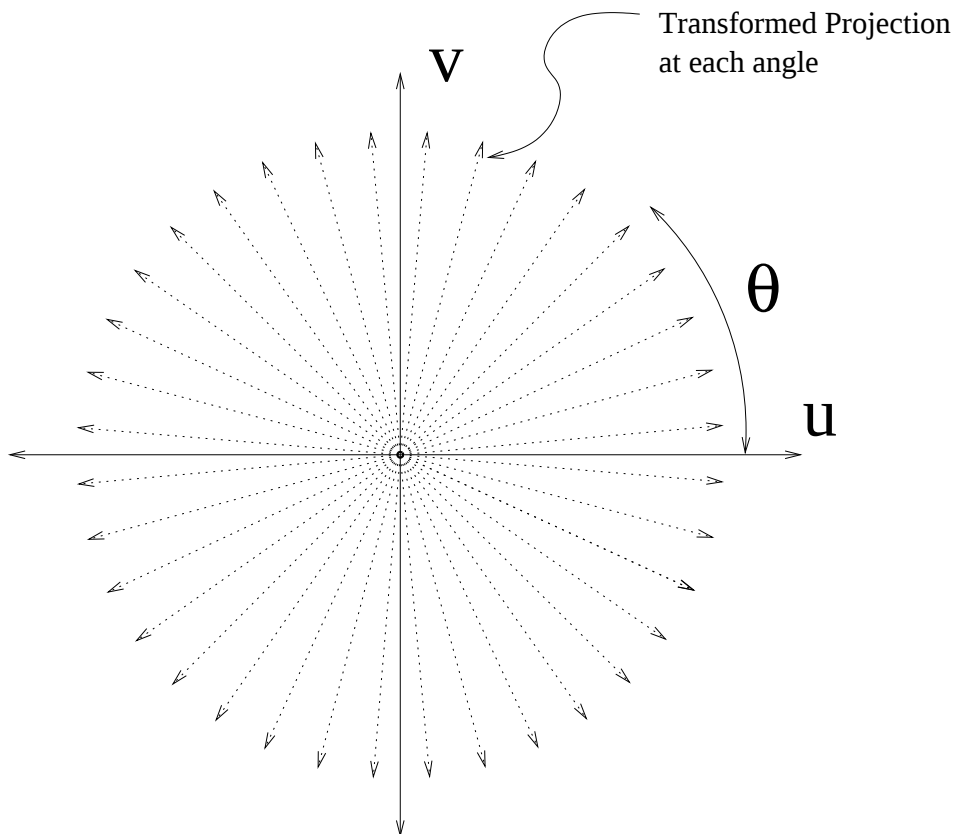
Then

$$\begin{aligned} P_0(\rho) &= \int_{-\infty}^{\infty} p_0(r) e^{-2\pi j r \rho} dr \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(r, y) dy \right] e^{-2\pi j r \rho} dr \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, y) e^{-2\pi j (r\rho + y0)} dr dy \\ &= F(\rho, 0) \end{aligned}$$

- By rotation property of CSFT, it must hold for any θ .

Inverse Radon Transform

- Physical systems measure $p_\theta(r)$.
- From these, we compute $P_\theta(\rho) = CTFT\{p_\theta(r)\}$.



- Next we take an inverse CSFT to form $f(x, y)$.

Problem: This requires polar to rectangular conversion.

Solution: Convolution backprojection

Convolution Back Projection (CBP) Algorithm

- In order to compute the inverse CSFT of $F(u, v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du dv = |\rho| d\theta d\rho$$

- This results in the expression

$$\begin{aligned} f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi j(xu+yv)} du dv \\ &= \int_{-\infty}^{\infty} \int_0^{\pi} P_{\theta}(\rho) e^{2\pi j(x\rho \cos(\theta)+y\rho \sin(\theta))} |\rho| d\theta d\rho \\ &= \int_0^{\pi} \left[\underbrace{\int_{-\infty}^{\infty} |\rho| P_{\theta}(\rho) e^{2\pi j\rho(x \cos(\theta)+y \sin(\theta))} d\rho}_{g_{\theta}(x \cos(\theta)+y \sin(\theta))} \right] d\theta \end{aligned}$$

- Then $g(t)$ is given by

$$\begin{aligned} g_{\theta}(t) &= \int_{-\infty}^{\infty} |\rho| P_{\theta}(\rho) e^{2\pi j\rho t} d\rho \\ &= CTFT^{-1} \{ |\rho| P_{\theta}(\rho) \} \\ &= h(t) * p_{\theta}(r) \end{aligned}$$

where $h(t) = CTF T^{-1} \{|\rho|\}$, and

$$f(x, y) = \int_0^\pi g_\theta (x \cos(\theta) + y \sin(\theta)) d\theta$$

Summary of CBP Algorithm

1. Measure projections $p_\theta(r)$.
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.
3. Back project filtered projections

$$f(x, y) = \int_0^\pi g_\theta (x \cos(\theta) + y \sin(\theta)) d\theta$$

A Closer Look at Projection Filter

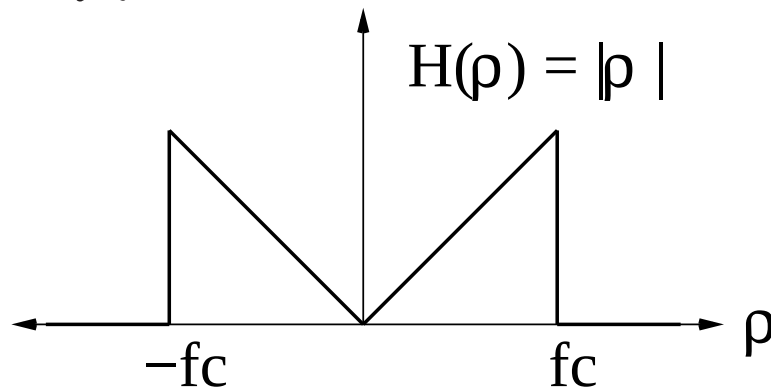
1. At each angle, projections are filtered.

$$g_{\theta}(r) = h(r) * p_{\theta}(r)$$

2. The frequency response of the filter is given by

$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to $|\rho| \leq f_c$ for some cut-off frequency f_c .



So

$$H(\rho) = f_c [\text{rect}(f/(2f_c)) - \Lambda(f/f_c)]$$

$$h(r) = f_c^2 [2\text{sinc}(t2f_c) - \text{sinc}^2(tf_c)]$$

A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^\pi b_\theta(x, y) d\theta$$

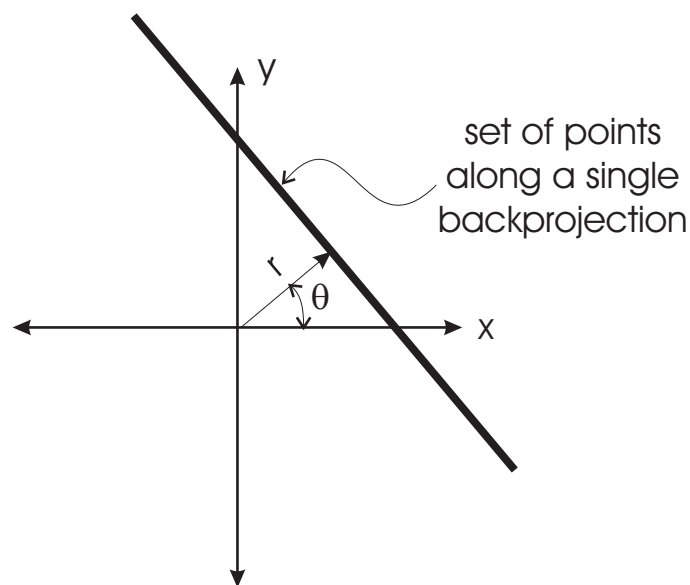
where

$$b_\theta(x, y) = g_\theta(x \cos(\theta) + y \sin(\theta))$$

- Consider the set of points (x, y) such that

$$r = x \cos(\theta) + y \sin(\theta)$$

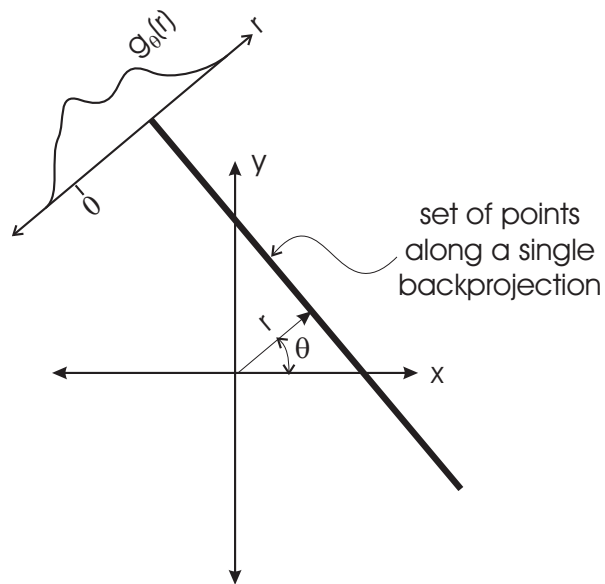
This set looks like



- Along this line $b_\theta(x, y) = g_\theta(r)$.

Back Projection Continued

- For each angle θ back projection is constant along lines of angle θ and takes on value $g_\theta(r)$.



- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to π .

$$f(x, y) = \int_0^\pi b_\theta(x, y) d\theta$$

$$\approx \frac{\pi}{M} \sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x, y)$$

- Back projection “smears” values of $g(r)$ back over image, and then adds smeared images for each angle.