

Filtered Random Processes

- Consider the 2-D linear system

$$Y(m, n) = h(m, n) * X(m, n)$$

where $X(m, n)$ is a 2-D wide sense stationary random process.

- It may be easily shown that

$$R_y(m, n) = h(m, n) * h(-m, -n) * R_x(m, n)$$

$$S_y(e^{j\mu}, e^{j\nu}) = |H(e^{j\mu}, e^{j\nu})|^2 S_x(e^{j\mu}, e^{j\nu})$$

2D Gaussian White Noise

- Definition:
 - $X(m, n)$ - independent identically distributed (i.i.d.) Gaussian random variables with distribution $N(0, 1)$.
- Then
 - $X(m, n)$ is wide sense stationary with

$$\mu(m, n) = E[X(m, n)] = 0$$

$$R_x(k, l) = E[X(0, 0)X(k, l)] = \delta(k, l)$$

$$\begin{aligned} S_x(e^{j\mu}, e^{j\nu}) &= DSFT \{R_x(k, l)\} \\ &= 1 \end{aligned}$$

Filtered White Noise

- Definitions:
 - $X(m, n)$ - independent identically distributed (i.i.d.) Gaussian random variables with distribution $N(0, 1)$.
 - $Y(m, n) = h(m, n) * X(m, n)$.
- Then
 - $Y(m, n)$ is wide sense stationary with

$$\begin{aligned}
 S_y(e^{j\mu}, e^{j\nu}) &= |H(e^{j\mu}, e^{j\nu})|^2 S_x(e^{j\mu}, e^{j\nu}) \\
 &= |H(e^{j\mu}, e^{j\nu})|^2 \cdot 1
 \end{aligned}$$

$$\begin{aligned}
 R_y(k, l) &= h(m, n) * h(-m, -n) \\
 &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m, n) h(m + k, n + l)
 \end{aligned}$$
 - $R_y(k, l)$ is the autocorrelation of $h(m, n)$ with itself.

Causal Prediction

- Y_s is a 2-D wide sense stationary zero mean Gaussian random process.
- Define
 - The past values are $Y_{<s} = \{Y_r : r < s\}$.
 - The minimum mean squared error (MMSE) predictor of Y_s given the past is

$$\hat{Y}_s = E[Y_s | Y_{<s}]$$

- The prediction error is $X_s = Y_s - \hat{Y}_s$.

Properties of Causal Predictors

- Fact 1: (WLOG, assume $r < s$.)

$$\begin{aligned}
 E[X_s X_r] &= E[E[X_s X_r | Y_{<s}]] \\
 &= E[E[(Y_s - \hat{Y}_s)(Y_r - \hat{Y}_r) | Y_{<s}]] \\
 &= E[E[(Y_s - \hat{Y}_s) | Y_{<s}](Y_r - \hat{Y}_r)] \\
 &= E[(E[Y_s | Y_{<s}] - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
 &= E[(\hat{Y}_s - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
 &= E[0(Y_r - \hat{Y}_r)] = 0
 \end{aligned}$$

- Fact 2: $\sigma^2 \triangleq E[X_s^2]$ is the prediction variance.
- Fact 3: The predictor must be space invariant and linear.

$$\hat{Y}_s = \sum_{r > (0,0)} h_r Y_{s-r}$$

Autoregressive (AR) Processes

- Definitions:
 - Y_s - 2-D wide sense stationary zero mean Gaussian random process.
 - h_s - MMSE linear predictor for Y_s .
 - $X_s = Y_s - h_s * Y_s$ - predictor error.
- If h_s is FIR, then Y_s is known as an autoregressive (AR) process.

Properties of AR Processes

- Remember that

$$X_s = Y_s - h_s * Y_s$$

- Then

- We know that

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Since X_s is white noise,

$$R_x(s) = \sigma^2 \delta(s)$$

$$S_x(e^{j\mu}, e^{j\nu}) = \sigma^2$$

- So the power spectrum of Y_s is given by

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\sigma^2}{|1 - H(e^{j\mu}, e^{j\nu})|^2}$$

Spectral Estimate Using AR Processes

- Compute MMSE linear predictor $\hat{h}_s$ for Y_s .
- Compute the prediction variance

$$\hat{\sigma}^2 = \frac{1}{|S|} \sum_{s \in S} |Y_s - h_s * Y_s|^2$$

where S is a finite set of points in plain, and $|S|$ is the number of points in S .

- Estimate the power spectrum

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\hat{\sigma}^2}{\left|1 - \hat{H}(e^{j\mu}, e^{j\nu})\right|^2}$$

- Can produce a more accurate estimate of the power spectrum.

Generating AR Processes

- Select a causal prediction filter h_s .
- Apply IIR filter to white noise random process X_s

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Y_s is sometimes referred to as a white noise driven process.
- Do linear FIR prediction filters $\hat{h}_s$ always form a stable IIR filter?
 - In 1-D, yes.
 - In 2-D, not always!
- Other problems:
 - Causal ordering of points may cause asymmetric artifacts in results.
 - Complexity increases rapidly with IIR filter order P .