EE 637 Final Exam

May 6, Spring 2022

Name: (2 pt) **Key**

Instructions:

- This is a 50 minute exam containing **three** problems.
- You may **only** use your brain and a pencil (or pen) and the included "Fact Sheet" to complete this exam.
- You **may not** use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You **may not** communicate with any person other than the official proctor during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

Q2 White Noise Screen (30 points)

Consider a 2D discrete-space image $f(i,j) \in [0,1]$ and an associated 2D white noise random screen $T(i,j)$ where for each i and j, $T(i,j)$ is an i.i.d. uniformly distributed random variable on the interval $[0,1]$. The the resulting binary halftone of f is then given by

$$B(i,j) = \begin{cases} 1 & \text{if } f(i,j) \geq T(i,j) \\ 0 & \text{if } f(i,j) < T(i,j) \end{cases}$$

Furthermore, define the display error as

$$D(i,j) = f(i,j) - B(i,j) \ ,$$

and assume that $f(i,j) = g$ for some scalar value $g \in [0,1]$.

Q2.1) (5 points) Is the random process $B(i,j)$ strict sense stationary? Prove your answer.

Q2.2) (5 points) Calculate the mean of the halftone given by

$$\mu = E\left[B(i,j)\right] \ .$$

Q2.3) (5 points) Calculate the variance of the halftone given by

$$\sigma^2 = E\left[(B(i,j) - \mu)^2\right] \ .$$

Q2.4) (5 points) Calculate the autocorrelation of the display error given by

$$R_D(m,n) = E\left[D(i,j)D(i+m,j+n)\right] \ .$$

Q2.5) (5 points) Calculate the power spectrum of the display error, $S_D(e^{j\mu}, e^{j\nu})$.

Q2.6) (5 points) What is the disadvantage of a white noise screen? Justify your answer based on the frequency content of the display error.

Solution:

Q 2.1 $B(i, j)$ is strict sense stationary.

This is true because the distribution of B is not a function of position. In order to see this, since white noise random screen is i.i.d. $T(i, j) \sim U[0, 1]$, we have that $P\{T(i, j) \leq t\} = t$ for $t \in [0, 1]$. Given $f(i, j) = g$ for some scalar value $g \in [0, 1]$, we can rewrite $B(i, j)$.

$$B(i,j) = \begin{cases} 1 & \text{if } g \geq T(i,j) \\ 0 & \text{if } g < T(i,j) \end{cases}$$

Therefore,

$$\begin{array}{ll} P\{B(i,j)=1\} = P\{T(i,j) \leq g\} = g. & \forall i,j \\ P\{B(i,j)=0\} = P\{T(i,j) > g\} = 1-g. & \forall i,j \end{array}$$

So therefore the CDF, F_B of $B(i, j)$ is given b

$$F(b) = \begin{cases} 0 & \text{if } b < 0 \\ 1-g & \text{if } 0 \geq b < 1 \\ 1 & \text{if } 1 \geq b \end{cases}$$

Then for any set of points, $s_1, \cdots, s_n$, we have that $\forall n \in \mathbb{N}^+$ and $\forall r, s_1, s_2, .., s_n \in \mathbb{N}^2$

$$\begin{aligned} & \mathbb{P}\{B_{r+s_1} \leq b_1, B_{r+s_2} \leq b_2, \cdots, X_{r+s_n} \leq b_n\} \\ = \; & F(b_1)F(b_2)\cdots F(b_n) \end{aligned}$$

So since the distribution does not depend on r, the B must be strict sense stationary.

Q 2.2 From Q2.1, we have

$$\begin{array}{ll} P\{B(i,j)=1\} = P\{T(i,j) \leq g\} = g. & \forall i,j \\ P\{B(i,j)=0\} = P\{T(i,j) > g\} = 1-g. & \forall i,j \end{array}$$

Therefore,

$$\mu = E\left[B(i,j)\right] = 1 \cdot g + 0 \cdot (1-g) = g \; .$$

Q 2.3

$$\begin{aligned} \sigma^2 = E\left[(B(i,j)-\mu)^2\right] &= E\left[(B(i,j)-g)^2\right] \\ &= E\left[B(i,j)^2 - 2B(i,j)g + g^2\right] \\ &= E\left[B(i,j)^2\right] - 2g^2 + g^2 \\ &= \left(1^2 g + 0^2(1-g)\right) - g^2 \\ &= g - g^2 = g(1-g) \end{aligned}$$

Q 2.4

$$
\begin{aligned}
R_D(m,n) &= E[D(i,j)D(i+m,j+n)] \\
&= E[(f(i,j) - B(i,j))((f(i+m,j+n) - B(i+m,j+n))] \\
&= E[(g - B(i,j))(g - B(i+m,j+n))] \\
&= E\left[g^2 - B(i,j)g - B(i+m,j+n)g + B(i,j)B(i+m,j+n)\right]
\end{aligned}
$$

If $(m,n) = (0,0)$,

$$
\begin{aligned}
R_D(m,n) &= E\left[g^2 - 2B(i,j)g + B(i,j)^2\right] \\
&= g^2 - 2g^2 + E\left[B(i,j)^2\right] \\
&= g(1-g)
\end{aligned}
$$

If $(m,n) \neq (0,0)$,

$$
\begin{aligned}
R_D(m,n) =& E\left[g^2\right] - gE[B(i,j)] - gE[B(i+m,j+n)] \\
& + E[B(i,j)B(i+m,j+n)] \\
=& g^2 - g^2 - g^2 + E[B(i,j)]E[B(i+m,j+n)] \\
=& g^2 - g^2 - g^2 + g^2 = 0
\end{aligned}
$$

Therefore,

$$
R_D(m,n) = g(1-g)\delta(m,n)
$$

Q 2.5 Calculate the DTFT of $R_D(m,n)$ $S_D(e^{j\mu}, e^{j\nu}) = g(1-g)$

Q 2.6 Since the power spectrum of the display error is a constant, which means that it has both low and high frequency error. The low frequency error will pass through the MTF of the human visual system and will appear as low frequency noise in the halftone.

Q3 Tomography (35 points)

Consider a 2D continuous-space image $f(x, y)$ with CSFT given by $F(u, v)$ and a forward projection given by

$$p_\theta(r) = \mathrm{FP}\{f(x, y)\}$$
$$= \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta))dz \ ,$$

and let $P_\theta(\rho)$ denote the CTFT of $p_\theta(r)$.

Q3.1) (5 points) Sketch a function $f(x, y)$ and the associated function $p_\theta(r)$ in a way that graphically illustrates their relationship.

Q3.2) (5 points) Give an explicit expression for the function $F(\rho\cos(\theta), \rho\sin(\theta))$ in terms of an integral of $p_\theta(r)$.

Q3.3) (5 points) Prove that if $F(u, v)$ is band limited to $\sqrt{u^2 + v^2} \leq f_c$, then $p_\theta(r)$ is also band limited to f_c.

Q3.4) (5 points) Compute the function $p_\theta(r) = \mathrm{FP}\{f(x, y)\}$ where

$$f(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

Q3.5) (5 points) Sketch the functions $p_\theta(r)$ and $f(x, y)$.

Q3.6) (5 points) Compute the function $\tilde{p}_\theta(r) = \mathrm{FP}\left\{\tilde{f}(x, y)\right\}$ where

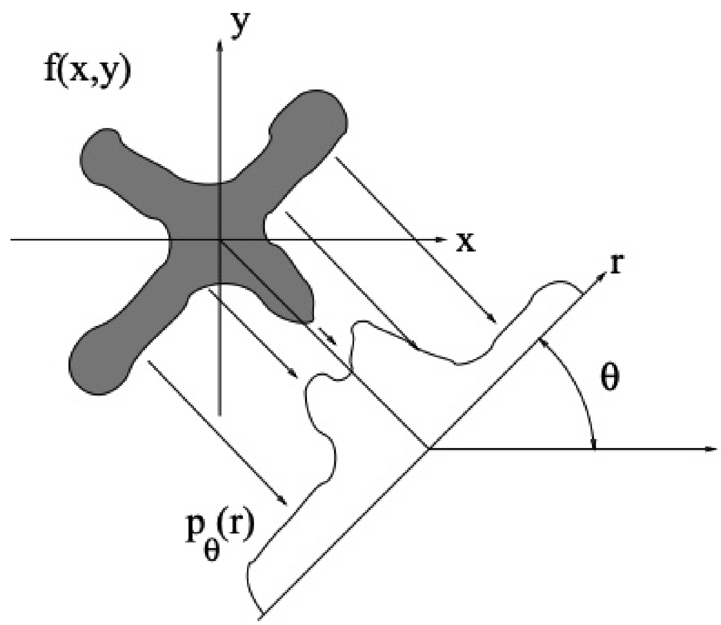
$$\tilde{f}(x, y) = f(x - x_o, y - y_o)$$

where x_o and y_o are constant offsets.

Q3.7) (5 points) Sketch the functions $\tilde{p}_\theta(r)$ and $\tilde{f}(x, y)$ for $x_o = y_o = 1$.

Solution:

Q 3.1



Q 3.2

$$
\begin{aligned}
F(\rho\cos\theta, \rho\sin\theta) &= P_\theta(\rho) \\
&= CTFT\{p_\theta(r)\} \\
&= \int_{-\infty}^{+\infty} p_\theta(r) e^{-j2\pi\rho r} dr
\end{aligned}
$$

Q 3.3 We know that

$$
\begin{aligned}
P_\theta(\rho) &= F(\rho\cos\theta, \rho\sin\theta) \\
&= F(u_o, v_o)
\end{aligned}
$$

where $u_o = \rho\cos\theta$ and $v_o = \rho\sin\theta$. In addition, we know that

$$\sqrt{u_0^2 + v_o^2} = \rho\sqrt{\cos^2(\theta) + \sin^2(\theta)} = \rho \ .$$

So consequently, when $\rho \geq f_c$, we know that $\sqrt{u_o^2 + v_o^2} \geq f_c$, and therefore, we have that

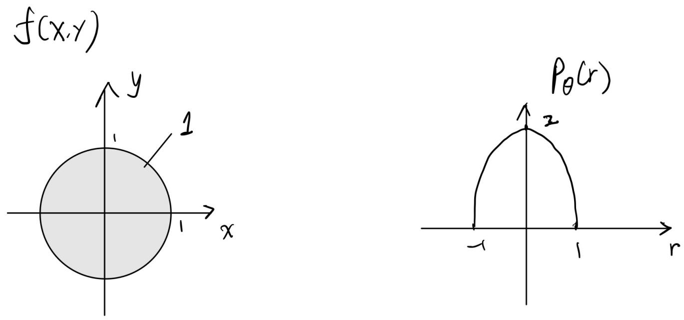
$$P_\theta(\rho) = F(u_o, v_o) = 0 \ ,$$

which means that $P_\theta(\rho)$ is bandlimited to $|\rho| \leq f_c$.

Q 3.4 If $f(x, y)$ is a circle with radius 1, then

$$p_\theta(r) = 2\sqrt{1 - r^2}\ \mathrm{rect}(r/2)\ .$$

Q 3.5



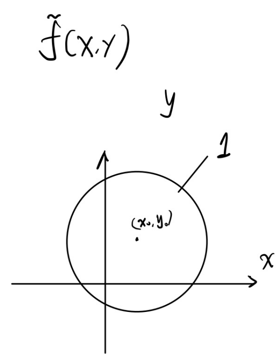
Q 3.6 Let $g(x, y) = f(x - x_0, y - y_0)$, $p^F_\theta(r) = FP\{f(x, y)\}$, $p^G_\theta(r) = FP\{g(x, y)\}$.

$$\begin{aligned}
p^G_\theta(r) &= \int_{-\infty}^{+\infty} g\left(A_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{+\infty} f\left(A_\theta \begin{bmatrix} r \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{+\infty} f\left(A_\theta \left(\begin{bmatrix} r \\ z \end{bmatrix} - A_\theta^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right)\right) dz \\
&= \int_{-\infty}^{+\infty} f\left(A_\theta \begin{bmatrix} r - (x_0 \cos\theta + y_0 \sin\theta) \\ z - (-x_0 \sin\theta + y_0 \cos\theta) \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{+\infty} f\left(A_\theta \begin{bmatrix} r - (x_0 \cos\theta + y_0 \sin\theta) \\ z' \end{bmatrix}\right) dz' \\
&= p^F_\theta\left(r - x_0 \cos\theta - y_0 \sin\theta\right)
\end{aligned}$$

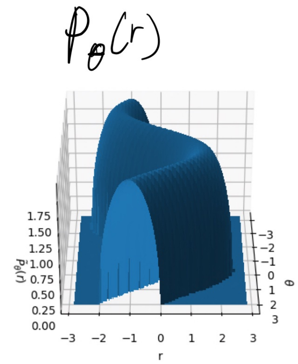
Therefore,

$$\tilde{p}_\theta(r) = 2\sqrt{1 - (r - x_0 \cos\theta - y_0 \sin\theta)^2}\ \mathrm{rect}\left((r - x_0 \cos\theta - y_0 \sin\theta)/2\right)$$

Q 3.7



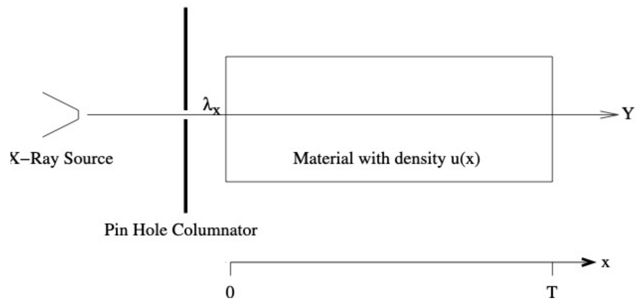
f̃(x,y)
y
1
(x₀,y₀)
x



P_θ(r)
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−3
−2
−1
0
1
2
3
r
θ

Q4 X-ray Transmission (20 points)

Consider an X-ray imaging system shown in the figure below.



Photons are emitted from an X-ray source and collimated by a pin hole in a lead shield. The collimated X-rays then pass in a straight line through an object of length T with density $u(x)$ where x is the depth into the object measured in units of cm, and $u(x)$ is measured in units of cm^{-1}. The expected number of photons at a depth x is given by

$$\lambda_x = E[Y_x]$$

where Y_x represents the number of photons that make it to depth x.

Q4.1) (5 points) Write a differential equation which describes the behavior of λ_x as a function of x.

Q4.2) (5 points) Solve the differential equation to form an expression for λ_x in terms of $u(x)$ and λ_0.

Q4.3) (5 points) Calculate an approximate expression for the integral of the density, $\int_0^T u(x)dx$, in terms of λ_0 and Y_T.

Q4.4) (5 points) Explain how the above result is used in computed tomographic imaging.

Solution:

Q 4.1

$$\frac{d\lambda_x}{dx} = -\lambda_x u(x)$$

Q 4.2

$$\lambda_x = \lambda_0 e^{-\int_0^x u(t)dt}$$

Q 4.3 From set $x = T$, then we have that

$$\lambda_T = \lambda_0 e^{-\int_0^T u(t)dt} \ .$$

So that

$$\frac{\lambda_T}{\lambda_0} = e^{-\int_0^T u(t)dt} \ .$$

Taking the negative log, we get that

$$-\log\left(\frac{\lambda_T}{\lambda_0}\right) = \int_0^T u(t)dt \ .$$

So finally, since we know that $\lambda_T = E[Y_T] \approx Y_T$, we have that

$$\int_0^T u(t)dt = -\log\left(\frac{\lambda_T}{\lambda_0}\right) \approx -\log\left(\frac{Y_T}{\lambda_0}\right) \ .$$

Q 4.4 Make two measurements.

- 1. Measure and calculate the expect number of photons without putting in an object: $\lambda_0 = E[Y_0] = Y_0$
- 2. Measure and calculate the expect number of photons with an object at every rotation angle, θ: $\lambda_{\theta,T} = E[Y_{\theta,T}] \approx Y_{\theta,T}$

With these two measurements at different rotation angles θ, we can obtain the forward projections $p_\theta(r) = \int_0^T u(t)dt = -\log\left(\frac{Y_{\theta,T}}{Y_0}\right)$ which are next used to calculate the density of an object in computed tomographic imaging.

Q5 Bilateral Filters (30 points)

Below we describe a bi-lateral image filter with input x_s and output y_s for $s = (s1, s2)$ where $s1$ and $s2$ are integers. The bilateral filter is specified by the following set of equations.

$$y_s = \sum_r w_{s,r} x_r \ ,$$

where

$$w_{w,r} = \frac{\tilde{w}_{s,r}}{\sum_k \tilde{w}_{s,k}} \tag{1}$$

$$\tilde{w}_{s,r} = \exp\left\{-\frac{1}{2\sigma_s^2}\|s-r\|^2\right\} \exp\left\{-\frac{1}{2\sigma_x^2}\|x_s - x_r\|^2\right\} \tag{2}$$

Q5.1) (5 points) What is the bilateral filter used for?

Q5.2) (5 points) Specify an advantage of the bilateral filter.

Specify a disadvantage of the bilateral filter.

Q5.3) (5 points) Explain why equation (1) above is required?

Q5.4) (5 points) Explain how σ_x should be selected?

Q5.5) (5 points) Is the bilateral filter linear? Justify your answer

Q5.6) (5 points) Is the bilateral filter space invariant? Justify your answer

Solution:

Q 5.1
The bilateral filter is used for noise reduction or smoothing. In addition, it preserves large sharp edges without blurring.

Q 5.2
Advantage: The filter will preserve edges when removing the noise.
Disadvantage: It is computationally expensive comparing with an LSI filter.

Q 5.3
Equation 1 is required to normalize all the weights in the filter, which maintains the DC gain of the filter to 1 so that the brightness of the image input will not change after filtering.

Q 5.4
The value of σ_x controls the assumed level of noise in the image. This is because it controls how much variation is assumed between adjacent pixels in the image. If the amount of variation is larger than σ_x, than then adjacent pixel will be down-weighted when computing the local average.

As σ_x increases, the bilateral filter becomes approximately Gaussian blur because second term will be almost a constant over the intensity interval covered by the image.

As σ_x decreases, the bilateral filter may lead to no smoothing occurring.

Q 5.5
No.
Notice that if we choose $x_s = \alpha_{small}\delta(s)$ where $0 < \alpha_{\text{small}} << 1$, then we have that

$$\exp\left\{-\frac{1}{2\sigma_x^2}\left\|x_s - x_r\right\|^2\right\} \approx 1$$

so that

$$\begin{aligned} w_{s,r} &\approx \frac{1}{z}\exp\left\{-\frac{1}{2\sigma_s^2}\|s-r\|^2\right\}\exp\left\{-\frac{1}{2\sigma_x^2}\left\|x_s - x_r\right\|^2\right\} \\ &= \frac{1}{z}\exp\left\{-\frac{1}{2\sigma_s^2}\|s-r\|^2\right\} \end{aligned}$$

where

$$z = \sum_{s\in\mathbb{Z}}\exp\left\{-\frac{1}{2\sigma_s^2}\|s\|^2\right\}$$

So then we have that,

$$y_s = \alpha_{small}\frac{\exp\left\{-\frac{1}{2\sigma_s^2}\|s\|^2\right\}}{\sum_{r\in\mathbb{Z}}\exp\left\{-\frac{1}{2\sigma_s^2}\|r\|^2\right\}}$$

Alternatively, if we choose $x_s = \alpha_{big}\delta(s)$ where $\alpha_{big} >> 1$, then we have that

$$F\left[x_s\right] = \alpha_{\text{big}}\,\delta(s)$$

If F is linear, then we must have that

$$\begin{aligned}
\alpha_{\text{big}}\,\delta(s) &= F\left[\alpha_{\text{big}}\,\delta(s)\right] = \alpha F[\delta(s)] \\
&= \frac{\alpha_{\text{big}}}{\alpha_{small}} F\left[\alpha_{small}\delta(s)\right] \\
&= \frac{\alpha_{big}}{\alpha_{small}} \alpha_{\text{small}}\, \frac{\exp\left\{-\frac{1}{2\sigma_s^2}\|s\|^2\right\}}{\sum_{r\in\mathbb{Z}} \exp\left\{-\frac{1}{2\sigma_s^2}\|r\|^2\right\}} \\
&= \alpha_{\text{big}}\, \frac{\exp\left\{-\frac{1}{2\sigma_s^2}\|s\|^2\right\}}{\sum_{r\in\mathbb{Z}} \exp\left\{-\frac{1}{2\sigma_s^2}\|r\|^2\right\}}
\end{aligned}$$

which is a contradiction, so F is not linear.

Q 5.6 Yes.
The bilateral filter is space invariant because the output y_s only depends on relative pixel value and their distance on grid.