Multivariate Gaussian Distribution

- Let $\mathbf{x}$ be a zero-mean random variable on $\mathbb{R}^p$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}} |R|^{-1/2} \exp\left\{ \frac{1}{2} \mathbf{x}^T R^{-1} \mathbf{x} \right\}$$

  where R is the $p \times p$ covariance matrix.

- The matrix R is a positive definite symmetric matrix, then

$$R = E \Lambda E^t$$

  where $E = [\mathbf{e}_1, \ldots, \mathbf{e}_p]$ is an orthonormal matrix of eigenvectors, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p)$ is a diagonal matrix of eigenvalues.

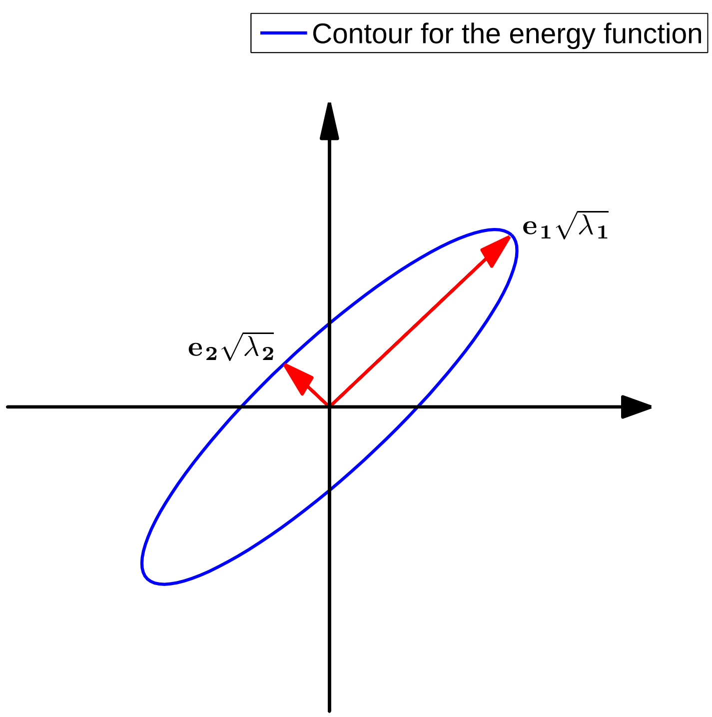
- Therefore, we have $E^t E = I$.

Contour for the Energy Function

- Intuitively, the energy function (square of the Mahalanobis distance)

$$f(\mathbf{x}) = \mathbf{x}^t R^{-1} \mathbf{x}$$

has contour plots shown here.



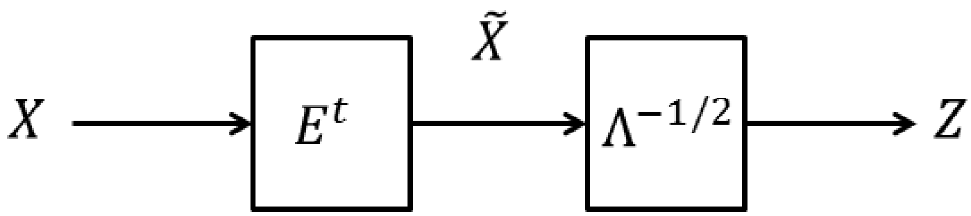
Gaussian Random Variable Decorrelation

- Consider $\tilde{\mathbf{x}} = E^t \mathbf{x}$, then

$$\begin{aligned} \mathbb{E}[\tilde{\mathbf{x}}\, \tilde{\mathbf{x}}^t] &= \mathbb{E}[E^t \mathbf{x}\, \mathbf{x}^t E] \\ &= E^t \mathbb{E}[\mathbf{x}\, \mathbf{x}^t] E \\ &= E^t R E \\ &= E^t E \Lambda E^t E \\ &= \Lambda \end{aligned}$$

- Therefore, the elements of $\tilde{\mathbf{x}}$ are uncorrelated with variance $\mathbb{E}[\tilde{x}_i^2] = \Lambda_{ii}$.
- The elements of $\tilde{\mathbf{x}}$ are independent, since $\tilde{\mathbf{x}}$, as the linear transform of $\mathbf{x}$, is Gaussian distributed.

Whitening Gaussian Random Variables



$$\mathbb{E}[Z\ Z] = I$$

- So E^t decorrelates $\mathbf{x}$, while $\Lambda^{-\frac{1}{2}}E^t$ whitens $\mathbf{x}$.

$$E^t\mathbf{x} = \begin{bmatrix} \mathbf{e}_1^t \\ \vdots \\ \mathbf{e}_p^t \end{bmatrix} \begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix}$$

- The eigenvectors $\mathbf{e}_k$, called eigen-signals, are basis vectors to represent the signal $\mathbf{x}$.
- If $\mathbf{x}$ represents an image, then the eigenvectors $\mathbf{e}_k$ are also called *eigenimages*.

Eigenimage Estimation

- Assume we have n training vectors $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, then

$$\begin{aligned} R_x =& E[\mathbf{x}_k \, \mathbf{x}_k^t] \\ =& E[XX^t]/n \\ \cong& XX^t/n \\ =& S \end{aligned}$$

where $S = \frac{1}{n} XX^t$ is the sample correlation matrix

$$S_{ij} = \frac{1}{n} \sum_{k=1}^{n} X_{ik} X_{jk}$$

- Decompose S as

$$S = \hat{E} \hat{\Lambda} \hat{E}^t$$

where $\hat{E}$ is an estimate of the eigenvectors, and $\hat{\Lambda}$ is an estimate of the eigenvalues.

- E could be very large, especially when X represents n images.

Singular Value Decomposition (SVD)

- For $n < p$ it looks like

$$\left[\begin{array}{c} X \\ \\ p \times n \\ \\ \end{array}\right] = \left[\begin{array}{c} U \\ \\ p \times n \\ \\ \end{array}\right] \left[\begin{array}{c} \Sigma \\ n \times n \end{array}\right] \left[\begin{array}{c} V^t \\ n \times n \end{array}\right]$$

  - The columns of U are orthonormal and called left hand singular vectors.
  - The columns of V are orthonormal and called right hand singular vectors.
  - Σ is diagonal matrix of singular values.

Eigenimage Estimation using SVD

- Notice that

$$\begin{aligned} XX^t &= U\Sigma V^t V\Sigma U^t \\ &= U\Sigma^2 U^t \end{aligned}$$

  So U is the set of the desired eigenvectors of XX^t.

- How to compute U?
  - Notice that $X^tX = V\Sigma^2V^t$ is a $n \times n$ matrix, and V contains eigenvectors of a much smaller matrix.
  - Algorithm
    * Find eigenvectors V of the matrix X^tX
    * Compute $XV = U\Sigma$, Then the columns of U are the normalized columns of XV, and Σ are the normalization factors.