

Continuous Space Fourier Transform (CSFT)

Forward CSFT:

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx dy$$

Inverse CSFT:

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} du dv$$

- Space coordinates:
 1. Usually, x is horizontal and y is vertical coordinate
 2. Usually, y points down
 3. Raster order - Television *scans* rapidly from left to right and more slowly from top to bottom.
- Frequency coordinates:
 1. u corresponds to horizontal frequency components (vertical strips).
 2. v corresponds to vertical frequency components (horizontal strips).

Useful Continuous Space Signal Definitions

$$\delta(x, y) \triangleq \delta(x) \delta(y)$$

$$\text{rect}(x, y) \triangleq \text{rect}(x) \text{rect}(y)$$

$$\text{sinc}(x, y) \triangleq \text{sinc}(x) \text{sinc}(y)$$

$$\text{circ}(x, y) \triangleq \text{rect}(\sqrt{x^2 + y^2})$$

- A 2-D function $f(x, y)$ is said to be **separable** if it is formed by the product of two 1-D functions.

$$f(x, y) = g(x) h(y)$$

$\text{rect}(x, y)$, $\text{sinc}(x, y)$, and $\delta(x, y)$ are separable functions.

- Is $\text{circ}(x, y)$ a separable function?

CSFT Properties Inherited from CTFT

- Some properties of the CSFT are very similar to corresponding CTFT properties.

Property	Space Domain Function	CSFT
Linearity	$af(x, y) + bg(x, y)$	$aF(u, v) + bG(u, v)$
Conjugation	$f^*(x, y)$	$F^*(-u, -v)$
Scaling	$f(ax, by)$	$\frac{1}{ ab }F(u/a, v/b)$
Shifting	$f(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0 + vy_0)}F(u, v)$
Modulation	$e^{j2\pi(u_0x + v_0y)}f(x, y)$	$F(u - u_0, v - v_0)$
Convolution	$f(x, y) * g(x, y)$	$F(u, v)G(u, v)$
Multiplication	$f(x, y)g(x, y)$	$F(u, v) * G(u, v)$
Duality	$F(x, y)$	$f(-u, -v)$

- Inner product property

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)g^*(x, y)dx dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)G^*(u, v)du dv
 \end{aligned}$$

Properties Specific to CSFT

- But some properties of the CSFT are quite unique to the 2-dimensional problem.

Property	Space Domain Function	CSFT
Separability	$f(x)g(y)$	$F(u)G(v)$
Rotation	$f\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$	$ \mathbf{A} ^{-1}F\left([u, v]\mathbf{A}^{-1}\right)$

Separability of CSFT

$$\begin{aligned} F(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx dy \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x, y) e^{-j2\pi ux} dx \right] e^{-j2\pi vy} dy \end{aligned}$$

Define the CTFT of $f(x, y)$ in the variable x

$$\tilde{F}(u, y) = \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi ux} dx$$

Then the CSFT may be computed as the CTFT of $\tilde{F}(u, y)$ in y

$$F(u, v) = \int_{-\infty}^{\infty} \tilde{F}(u, y) e^{-j2\pi vy} dy$$

- Comment: 2-D CSFT can be computed as two 1-D CTFT's.

CSFT of Separable Functions

Let

$$\begin{aligned} g(t) &\stackrel{CTFT}{\Leftrightarrow} G(f) \\ h(t) &\stackrel{CTFT}{\Leftrightarrow} H(f) \end{aligned}$$

Then

$$g(x)h(y) \stackrel{CSFT}{\Leftrightarrow} G(u)H(v)$$

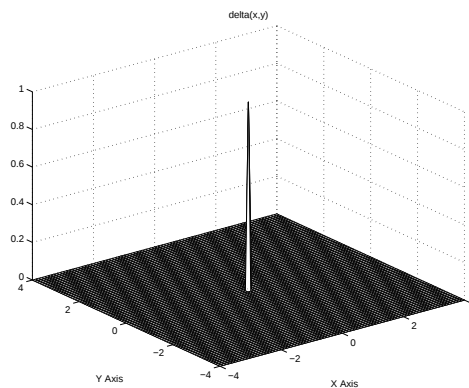
Proof:

$$\begin{aligned} F(u, v) &= CSFT \{g(x) h(y)\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) h(y) e^{-j2\pi(ux+vy)} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) h(y) e^{-j2\pi ux} e^{-j2\pi vy} dx dy \\ &= \left[\int_{-\infty}^{\infty} g(x) e^{-j2\pi ux} dx \right] \left[\int_{-\infty}^{\infty} h(y) e^{-j2\pi vy} dy \right] \\ &= G(u)H(v) \end{aligned}$$

Useful CSFT Transform Pairs

- 2-D delta function:

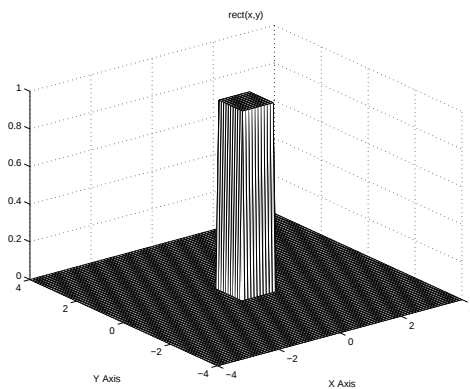
$$\begin{aligned}
 CSFT \{ \delta(x, y) \} &= CSFT \{ \delta(x) \delta(y) \} \\
 &= CTFT \{ \delta(x) \} \cdot CTFT \{ \delta(y) \} \\
 &= 1 \cdot 1 = 1
 \end{aligned}$$



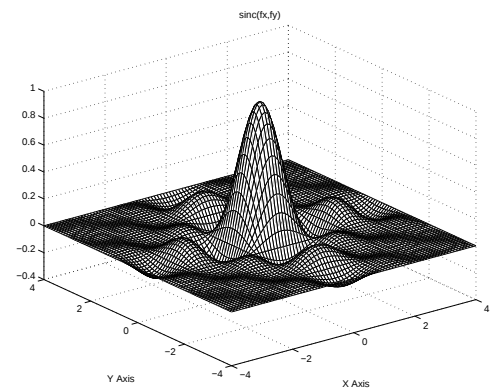
$\Rightarrow 1$

- 2-D rect function:

$$\begin{aligned}
 CSFT \{ \text{rect}(x, y) \} &= CSFT \{ \text{rect}(x) \text{rect}(y) \} \\
 &= CTFT \{ \text{rect}(x) \} \cdot CTFT \{ \text{rect}(y) \} \\
 &= \text{sinc}(u) \text{sinc}(v) \\
 &= \text{sinc}(u, v)
 \end{aligned}$$



$\Rightarrow$



Rotated Functions

- Let the matrix $\mathbf{A}$ be an orthonormal rotation of angle θ

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Because $\mathbf{A}$ is an orthonormal transform

$$|\mathbf{A}| = 1$$

$$\mathbf{A}^{-1} = \mathbf{A}^t$$

- Then the CSFT of the function $g\left(\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}\right)$ is given by

$$\begin{aligned} CSFT \left\{ g\left(\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}\right) \right\} &= |\mathbf{A}|^{-1} G([u, v] \mathbf{A}^{-1}) \\ &= |\mathbf{A}|^{-1} G([u, v] \mathbf{A}^t) \\ &= G\left(\mathbf{A} \begin{bmatrix} u \\ v \end{bmatrix}\right) \end{aligned}$$

- So we have

$$g\left(\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}\right) \stackrel{CSFT}{\Leftrightarrow} G\left(\mathbf{A} \begin{bmatrix} u \\ v \end{bmatrix}\right)$$

Rotated Rect Function

- Rotated 2-D rect function:

$$\text{rect} \left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}} \right) = \text{rect} \left(\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

where

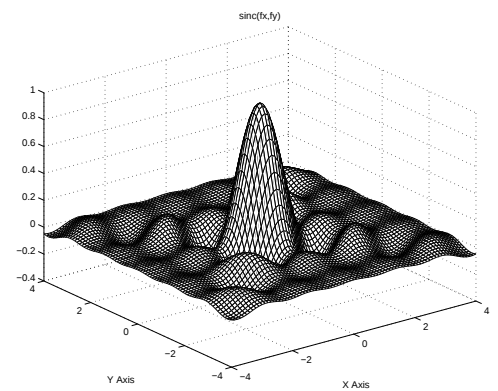
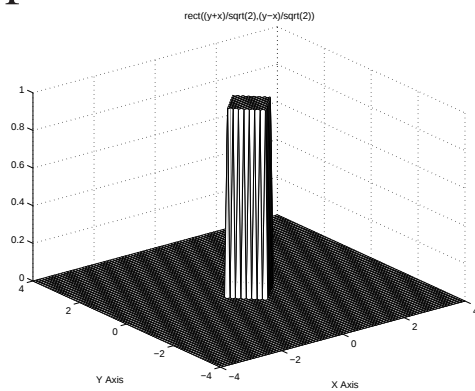
$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- $\mathbf{A}$ is a 45° rotation, so it is an orthonormal transform

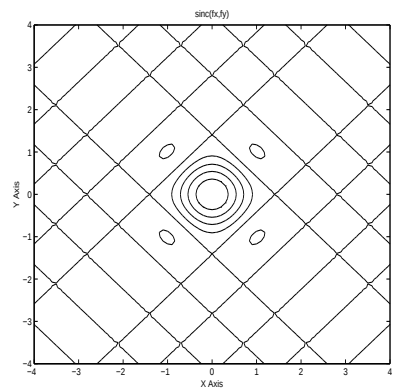
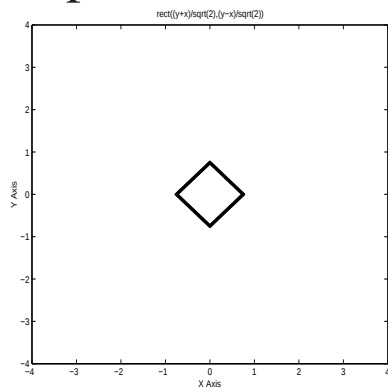
$$\begin{aligned} CSFT \left\{ \text{rect} \left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}} \right) \right\} &= CSFT \left\{ \text{rect} \left(\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \right) \right\} \\ &= \text{sinc} \left(\mathbf{A} \begin{bmatrix} u \\ v \end{bmatrix} \right) \\ &= \text{sinc} \left(\frac{v+u}{\sqrt{2}}, \frac{v-u}{\sqrt{2}} \right) \end{aligned}$$

Rotated 2-D Rect and Sinc Transform Pairs

- Mesh plot



- Contour plot



More Useful CSFT Transform Pairs

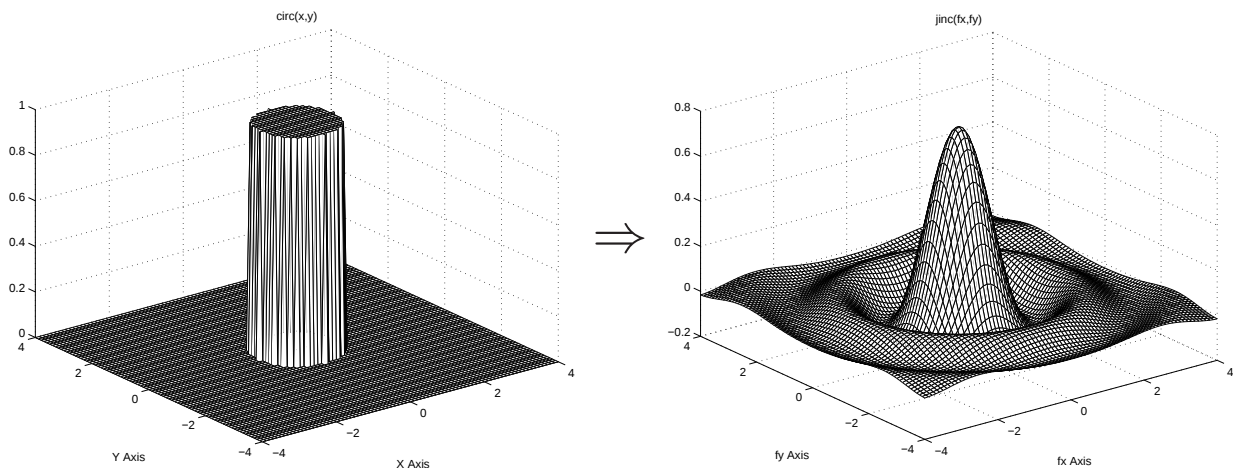
- Circ function:

$$CSFT \{ \text{circ}(x, y) \} = \text{jinc}(u, v)$$

where

$$\text{jinc}(u, v) = \frac{J_1 \left(\pi \sqrt{u^2 + v^2} \right)}{2\sqrt{u^2 + v^2}}$$

and $J_1(r)$ is the Bessel function of the first kind order 1.



- Notice that both functions are circularly symmetric

CSFT of a Plane Wave

- Consider an impulse in the 2-D frequency domain.

$$F(u, v) = \delta(u - u_o, v - v_o)$$

- Its inverse transform is a 2-D plane wave.

$$\begin{aligned} f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u - u_o, v - v_o) e^{j2\pi(ux+vy)} du dv \\ &= e^{j2\pi(u_o x + v_o y)} \end{aligned}$$

- We know that

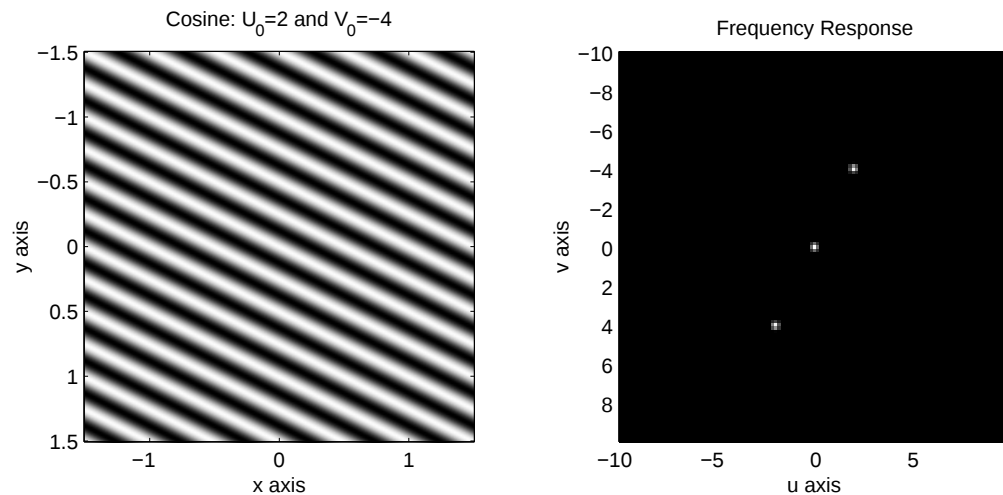
$$\cos(2\pi(u_o x + v_o y)) = \frac{1}{2} \left[e^{j2\pi(u_o x + v_o y)} + e^{-j2\pi(u_o x + v_o y)} \right]$$

- So we have that

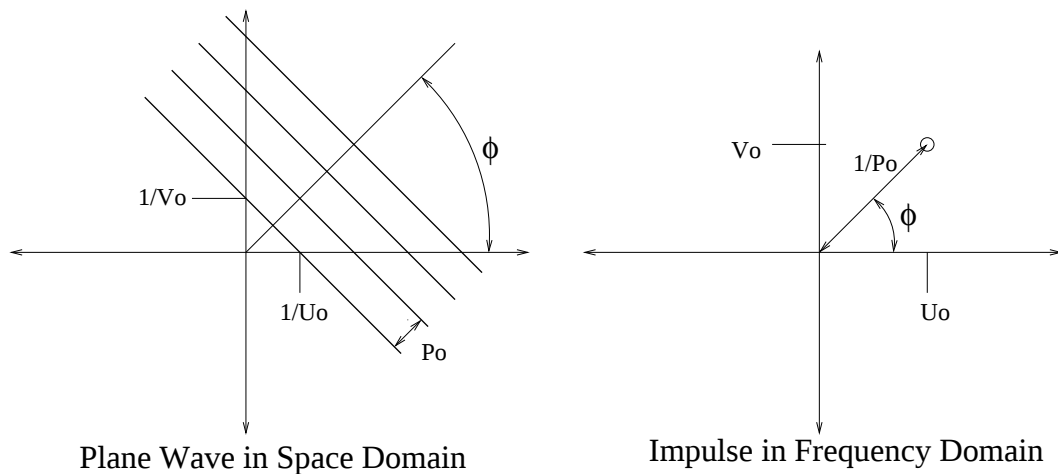
$$\overset{CSFT}{\Leftrightarrow} \frac{\cos(2\pi(u_o x + v_o y))}{2} [\delta(u - u_o, v - v_o) + \delta(u + u_o, v + v_o)]$$

2-D Plane Wave Example

- Example transform pair computed with Matlab ¹



- Graphical representation of space-frequency domain

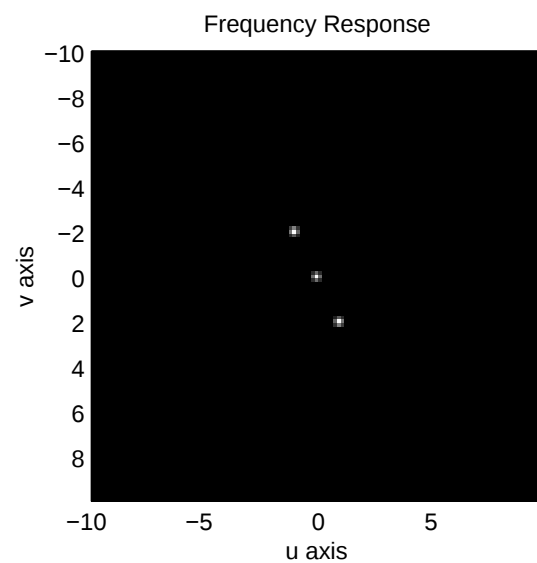
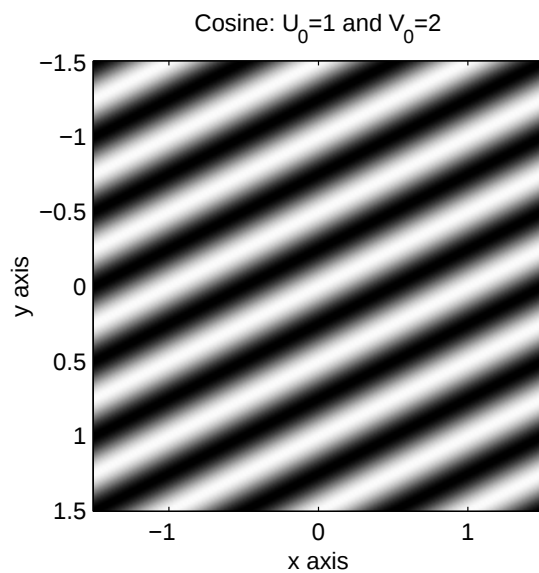
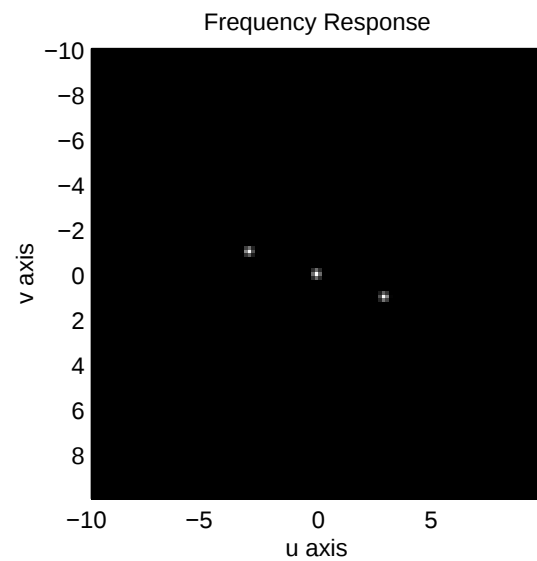
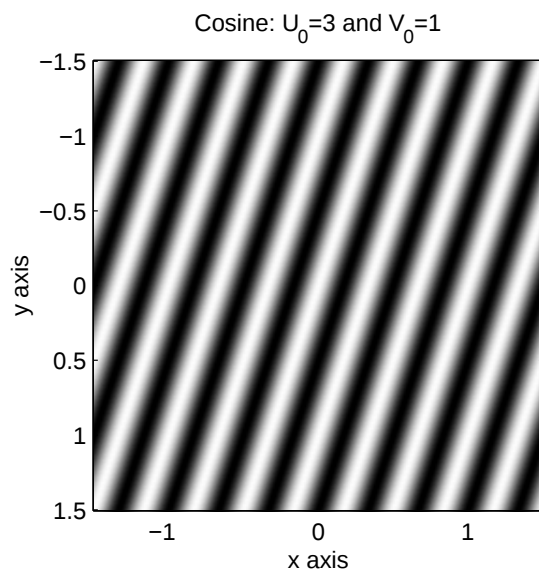


$$- 1/P_0 = \sqrt{V_0^2 + U_0^2}$$

- Rotations in space and frequency domains are the same.

¹ $f(x, y) = \cos(u_0 x + v_0 y) + 0.5$

More Examples 2-D Plane Waves



More Examples 2-D Plane Waves

