EE 637 Final Exam
May 4, Spring 2020

Q1.

Rules: I understand that this is an open book exam that shall be done within the time allotted by Gradescope. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

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Name: **Key**

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

Q2 Power Spectrum

Let $x(n)$ be the input and $y(n)$ be the output for a discrete-time LTI system with impulse response

$$h(n) = \delta(n) - \frac{1}{2}(\delta(n-1) + \delta(n+1))$$

Furthermore, assume that $X(n)$ are i.i.d. $N(0, \sigma^2)$ random variables.

Q2.1
Calculate the autocovariance, $R_x(n)$, of $x(n)$.

Q2.2
Calculate the power spectrum, $S_x(e^{j\omega})$, of $x(n)$.

Q2.3
Calculate the power spectrum, $S_y(e^{j\omega})$, of $y(n)$.

Q2.4
Calculate the autocovariance, $R_y(n)$, of $y(n)$.

Solution:

Q2.1

$$R_x(n) = \sigma^2 \delta(n)$$

Q2.2

$$S_x(e^{j\omega}) = \sigma^2$$

Q2.3
The DTFT of $h(n)$ is given by

$$\begin{aligned} H(e^{j\omega}) &= 1 - \frac{1}{2}\left(e^{+j\omega} + e^{-j\omega}\right) \\ &= 1 - \cos(\omega) \end{aligned}$$

$$\begin{aligned} S_y(e^{j\omega}) &= \sigma^2 |H(e^{j\omega})|^2 \\ &= \sigma^2 |1 - \cos(\omega)|^2 \end{aligned}$$

Q2.4

$$\begin{aligned} R_y(n) &= \sigma^2 h(-n) * h(n) \\ &= \sigma^2 \left[\frac{3}{2}\delta(n) - (\delta(n-1) + \delta(n+1)) + \frac{1}{4}(\delta(n-2) + \delta(n+2))\right] \end{aligned}$$

Q3 Bilateral Filter Sampling

A bilateral filter is a discrete-space system, $y = F[x]$, with 2D input x_s and 2D output y_s where $s \in \mathbb{Z}^2$ and $\mathbb{Z}$ denotes the integers. More specifically, the input/output relationship is defined by

$$y_s = \sum_{r \in \mathbb{Z}^2} w_{s,r} x_r \ ,$$

where we define the kernel $w_{s,r}$ using the following two steps

$$\tilde{w}_{s,r} = \exp\left\{ -\frac{1}{2\sigma_1^2} \|s - r\|^2 \right\} \exp\left\{ -\frac{1}{2\sigma_2^2} \|x_s - x_r\|^2 \right\}$$

and

$$w_{s,r} = \frac{\tilde{w}_{s,r}}{\sum_{r' \in \mathbb{Z}^2} \tilde{w}_{s,r'}} \ ,$$

where σ^2 and σ_g^2 are filter parameters.
For subproblems 3.2 to 3.4 also assume that

$$x(m,n) = u(n) + w(m,n)$$

where $m, n \in \mathbb{Z}$, $u(n)$ is a discrete-time step function, and $w(m,n)$ are i.i.d. $\sim N(0, \sigma^2)$ where $\sigma = 1/10$.

Q3.1
Is this filter linear? Justify your answer.

Q3.2
What is a good choice for the filter parameter σ_2? Justify your answer.

Q3.3
As the value of σ_1 becomes larger, what happens to the noise in the output, y_s?

Q3.4
What is the advantage of this filter over a conventional LSI filter?

Solution:

Q3.1
No.
Notice that if we choose $x_s = \alpha_{small}\delta(s)$ where $0 < \alpha_{small} << 1$, then we have that

$$\exp\left\{ -\frac{1}{2\sigma_2^2} \|x_s - x_r\|^2 \right\} \approx 1 \ ,$$

so that

$$\begin{aligned} w_{s,r} &\approx \frac{1}{z} \exp\left\{ -\frac{1}{2\sigma_1^2} \|s - r\|^2 \right\} \exp\left\{ -\frac{1}{2\sigma_2^2} \|x_s - x_r\|^2 \right\} \\ &= \frac{1}{z} \exp\left\{ -\frac{1}{2\sigma_1^2} \|s - r\|^2 \right\} \end{aligned}$$

where

$$z = \sum_{s \in \mathbb{Z}} \exp\left\{ -\frac{1}{2\sigma_1^2} \|s\|^2 \right\}$$

So then we have that for α small,

$$y_s = \alpha_{small} \frac{\exp\left\{ -\frac{1}{2\sigma_1^2} \|s\|^2 \right\}}{\sum_{r\in\mathbb{Z}} \exp\left\{ -\frac{1}{2\sigma_1^2} \|r\|^2 \right\}}$$

Alternatively, if we choose $x_s = \alpha_{big}\delta(s)$ where $\alpha_{big} >> 1$, then we have that

$$F[x_s] = \alpha_{big}\,\delta(s)$$

If F is linear, then we must have that

$$\begin{aligned}
\alpha_{big}\,\delta(s) &= F\left[\alpha_{big}\delta(s)\right] = \alpha F\left[\delta(s)\right] \\
&= \frac{\alpha_{big}}{\alpha_{small}} F\left[\alpha_{small}\delta(s)\right] \\
&= \frac{\alpha_{big}}{\alpha_{small}} \alpha_{small} \frac{\exp\left\{ -\frac{1}{2\sigma_1^2} \|s\|^2 \right\}}{\sum_{r\in\mathbb{Z}} \exp\left\{ -\frac{1}{2\sigma_1^2} \|r\|^2 \right\}} \\
&= \alpha_{big} \frac{\exp\left\{ -\frac{1}{2\sigma_1^2} \|s\|^2 \right\}}{\sum_{r\in\mathbb{Z}} \exp\left\{ -\frac{1}{2\sigma_1^2} \|r\|^2 \right\}}
\end{aligned}$$

which is a contradiction, so F is not linear.

Q3.2

The image consists of dark pixels 0, and light pixels 1. So we would like to choose σ_2 so that pixels of the same type are assigned large values of $w_{s,r}$, and pixels of different type are assigned small values of $w_{s,r}$.

Any good choice of σ_2 should have that $\sigma_2 \geq \sigma = 0.1$ and $\sigma_2 << 1$ since this will included pixels in the same gray level, and reject pixels in the different gray levels.

So for example, a good choice is $\sigma_2 = 1/3$ since this meets both criteria.

Q3.3

As σ_2 becomes larger, the each region is filtered with a larger spatial psf. Therefore, the noise variance goes down and the spatial correlation of the noise goes down.

Q3.4

This filter preserves edges will removing noise. A conventional LSI filter tends to blur edges when it removes noise.

Q4 Convolutional Neural Network

Consider the following convolutional neural network (CNN) with an RGB input image and an RGB output image where each layer is formed by a convolution followed by a ReLU operation. Furthermore, we represent the first layer by

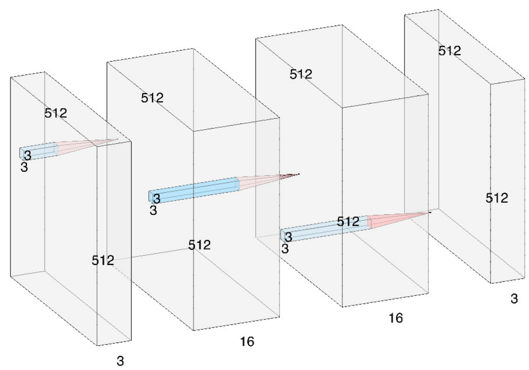
$$z_1 = \sigma(W_0 \cdot z_0 + b_0) \ ,$$

the second layer by

$$z_2 = \sigma(W_1 \cdot z_1 + b_1) \ ,$$

and the third layer by

$$z_3 = \sigma(W_2 \cdot z_2 + b_2) \ .$$



Q4.1
What are the ranks and dimensions of the first layer parameters W_0 and b_0? (Hint: A 2D array of size 10×10 has rank 2 and dimensions 10×10.)

Q4.2
What are the ranks and dimensions of the second layer parameters W_1 and b_1?

Q4.3
What are the ranks and dimensions of the third layer parameters W_2 and b_2?

Q4.4
What are the total number of parameters in the CNN model?

Q4.5
What is the advantage of this CNN model over a model with the same number of layers in which each layer is a fully connected network?

Solution:

Q4.1
W_0 has rank 4 and dimensions $3 \times 16 \times (3 \times 3)$. b_0 has rank 1 and dimension 16.
Q4.2
W_1 has rank 4 and dimensions $16 \times 16 \times (3 \times 3)$. b_1 has rank 1 and dimension 16.
Q4.3

W_1 has rank 4 and dimensions $16 \times 3 \times (3 \times 3)$. b_1 has rank 1 and dimension 3.
Q4.4
total number of parameters =

$$(3 \times 16 \times 3 \times 3) + 16 + (16 \times 16 \times 3 \times 3) + 16 + (16 \times 3 \times 3 \times 3) + 3 = 3,203$$

Q4.5
The CNN has the following potential advantages:

- The CNN has a small faction of the number of parameters.
- Due to the dramatic reduction in parameters, both training and inference are much faster.
- It results in a space-invariant operator which is desirable for some applications.

Q5 Halftoning

Let $x(m,n) \in [0,1]$ be 2D discrete-space image, and let $T(m,n) \in [0,1]$ be a 2D set of thresholds such that $T(m,n)$ are i.i.d. random variables that are uniformly distributed on the interval $[0,1]$. Furthermore, let

$$b(m,n) = \begin{cases} 1 & \text{if } x(m,n) \geq T(m,n) \\ 0 & \text{otherwise} \end{cases} .$$

Furthermore, define display error as

$$\epsilon(m,n) = b(m,n) - x(m,n) .$$

Q5.1
Assuming that $x(m,n) = g$, calculate the expected binary output

$$E\left[b(m,n)\right] .$$

Q5.2
Assuming that $x(m,n) = g$, calculate the expected display error

$$E\left[\epsilon(m,n)\right] .$$

Q5.3
Assuming that $x(m,n) = g$, calculate the variance of the display error

$$E\left[\left(\epsilon(m,n) - E\left[\epsilon(m,n)\right]\right)^2\right] .$$

Q5.4
Assuming that $x(m,n) = g$, calculate the display error auto-covariance given by

$$R(k,l) = E\left[\left(\epsilon(m,n) - E\left[\epsilon(m,n)\right]\right)\left(\epsilon(m+k,n+l) - E\left[\epsilon(m+k,n+l)\right]\right)\right] .$$

Q5.5
Assuming that $x(m,n) = g$, calculate the display error power spectrum $S(\mu,\nu)$.

Solution:

Q5.1
Notice that

$$P\{x(m,n) \geq T(m,n)\} = P\{g \geq T(m,n)\} = P\{T(m,n) \leq g\} = g$$

So we have that

$$E\left[b(m,n)\right] = 1g + 0(1-g) = g$$

Q5.2

$$E\left[\epsilon(m,n)\right] = E\left[b(m,n) - x(m,n)\right] = E\left[b(m,n) - g\right] = E\left[b(m,n)\right] - g = g - g = 0$$

Q5.3
Since $E\left[\epsilon(m,n)\right] = 0$, we have that

$$\begin{aligned} Variance &= E\left[(\epsilon(m,n))^2\right] \\ &= (1-g)^2 P\{x(m,n) \geq T(m,n)\} + (0-g)^2\left(1 - P\{x(m,n) \geq T(m,n)\}\right) \\ &= g(1-g) \end{aligned}$$

Q5.4
Since the $T(m,n)$ are i.i.d. and $E\left[\epsilon(m,n)\right] = 0$, we have that

$$R(k,l) = g(1-g)\delta(k,l)$$

Q5.5

$$S(\mu,\nu) = g(1-g)$$