Types of Coding

- Source Coding - Code data to more efficiently represent the information
  - Reduces "size" of data
  - Analog - Encode analog source data into a binary format
  - Digital - Reduce the "size" of digital source data
- Channel Coding - Code data for transmition over a noisy communication channel
  - Increases "size" of data
  - Digital - add redundancy to identify and correct errors
  - Analog - represent digital values by analog signals
- Complete "Information Theory" was developed by Claude Shannon

Digital Image Coding

- Images from a 6 MPixel digital cammera are 18 MBytes each
- Input and output images are digital
- Output image must be smaller (i.e. $\approx$ 500 kBytes)
- This is a digital source coding problem

Two Types of Source (Image) Coding

- Lossless coding (entropy coding)
  - Data can be decoded to form exactly the same bits
  - Used in “zip”
  - Can only achieve moderate compression (e.g. 2:1 - 3:1) for natural images
  - Can be important in certain applications such as medical imaging
- Lossly source coding
  - Decompressed image is visually similar, but has been changed
  - Used in “JPEG” and “MPEG”
  - Can achieve much greater compression (e.g. 20:1 - 40:1) for natural images
  - Uses entropy coding

Entropy

- Let X be a random variables taking values in the set $\{0, \cdots, M-1\}$ such that

$$p_i = P\{X = i\}$$

- Then we define the entropy of X as

$$\begin{aligned} H(X) &= -\sum_{i=0}^{M-1} p_i \log_2 p_i \\ &= -E\left[\log_2 p_X\right] \end{aligned}$$

$H(X)$ has units of bits

Conditional Entropy and Mutual Information

- Let (X, Y) be a random variables taking values in the set $\{0, \cdots, M-1\}^2$ such that

$$p(i,j) \; = \; P\{X = i, Y = j\}$$

$$p(i|j) \; = \; \frac{p(i,j)}{\sum_{k=0}^{M-1} p(k,j)}$$

- Then we define the conditional entropy of X given Y as

$$\begin{aligned} H(X|Y) \; &= \; -\sum_{i=0}^{M-1} \sum_{j=0}^{M-1} p(i,j) \log_2 p(i|j) \\ &= \; -E\left[\log_2 p(X|Y)\right] \end{aligned}$$

- The mutual information between X and Y is given by

$$I(X;Y) = H(X) - H(X|Y)$$

  The mutual information is the reduction in uncertainty of X given Y.

Entropy (Lossless) Coding of a Sequence

- Let X_n be an i.i.d. sequence of random variables taking values in the set $\{0, \cdots, M-1\}$ such that

$$P\{X_n = m\} = p_m$$

  – X_n for each n is known as a symbol

- How do we represent X_n with a minimum number of bits per symbol?

A Code

- **Definition:** A code is a mapping from the discrete set of symbols $\{0, \cdots, M-1\}$ to finite binary sequences
  - For each symbol, m their is a corresponding finite binary sequence σ_m
  - $|\sigma_m|$ is the length of the binary sequence
- Expected number of bits per symbol (bit rate)

$$\begin{aligned}\bar{n} &= E\left[|\sigma_{X_n}|\right] \\ &= \sum_{m=0}^{M-1} |\sigma_m| \, p_m\end{aligned}$$

- Example for $M = 4$

m	$\rightarrow$	σ_m	$\lvert\sigma_m\rvert$
0	$\rightarrow$	$0\,1$	2
1	$\rightarrow$	$1\,0$	2
2	$\rightarrow$	0	1
3	$\rightarrow$	$1\,0\,0\,1\,0\,0$	6

  - Encoded bit stream

$$(0, 2, 1, 3, 2) \rightarrow (01|0|10|100100|0)$$

Fixed versus Variable Length Codes

- Fixed Length Code - $|\sigma_m|$ is constant for all m
- Variable Length Code - $|\sigma_m|$ varies with m
- Problem
  - Variable length codes may not be uniquely decodable
  - Example: Using code from previous page

$$(6) \rightarrow (100100)$$

$$(1, 0, 2, 2) \rightarrow (10|01|0|0)$$

  - Different symbol sequences can yield the same code

- **Definition:** A code is *Uniquely Decodable* if there exists only a single unique decoding of each coded sequence.
- **Definition:** A *Prefix Code* is a specific type of uniquely decodable code in which no code is a prefix of another code.

Lower Bound on Bit Rate

- **Theorem:** Let C be a uniquely decodable code for the i.i.d. symbol sequence X_n. Then the mean code length is greater than $H(X_n)$.

$$\begin{aligned}
\bar{n} &\stackrel{\triangle}{=} E\left[|\sigma_{X_n}|\right] \\
&= \sum_{m=0}^{M-1} |\sigma_m| \, p_m \\
&\geq H(X_n)
\end{aligned}$$

- Question: Can we achieve this bound?

- Answer: Yes! Constructive proof using Huffman codes

Huffman Codes

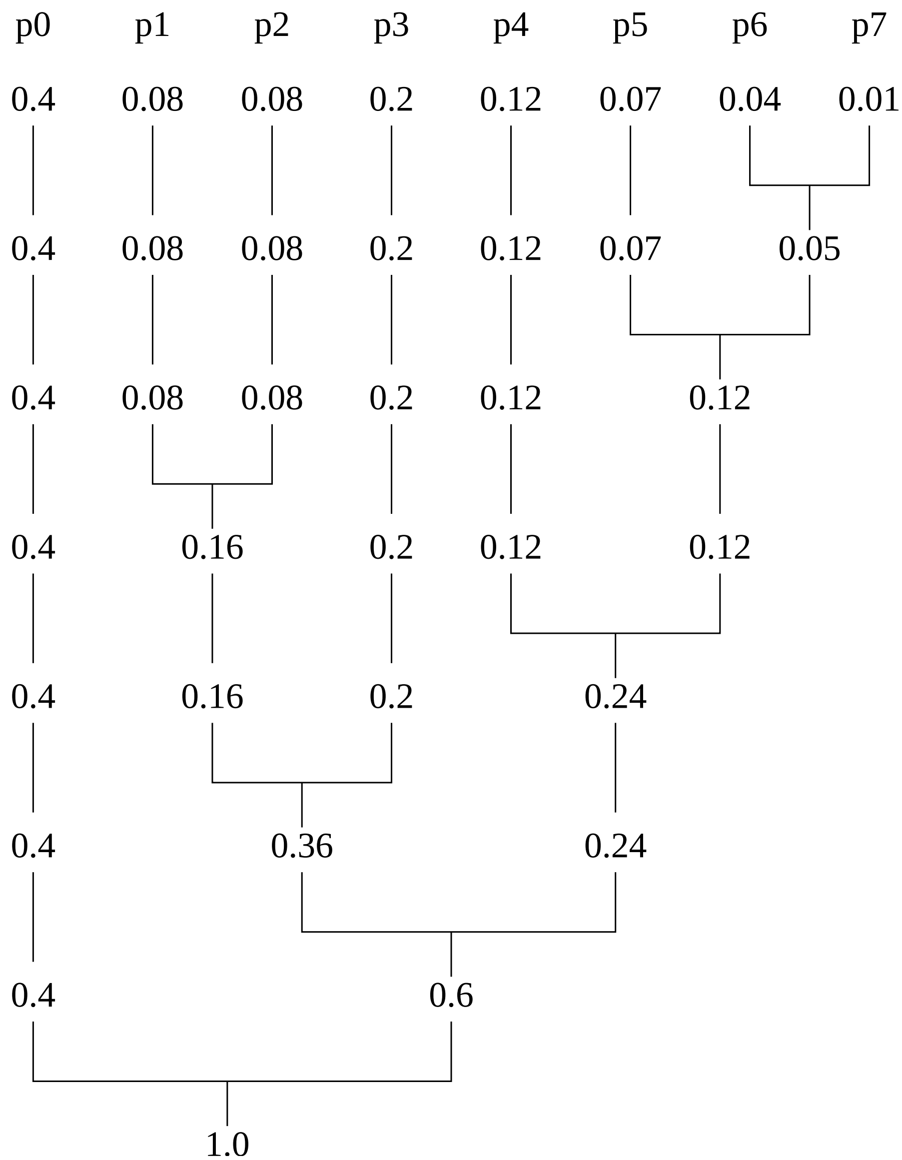
- Variable length prefix code $\Rightarrow$ Uniquely decodable
- Basic idea:
  - Low probability symbols $\Rightarrow$ Long codes
  - High probability symbols $\Rightarrow$ short codes
- Basic algorithm:
  - Low probability symbols $\Rightarrow$ Long codes
  - High probability symbols $\Rightarrow$ short codes

Huffman Coding Algorithm

1. Initialize list of probabilities with the probability of each symbol
2. Search list of probabilities for two smallest probabilities, p_{k*} and p_{l*}.
3. Add two smallest probabilities to form a new probability, $p_m = p_{k*} + p_{l*}$.
4. Remove p_{k*} and p_{l*} from the list.
5. Add p_m to the list.
6. Go to step 2 until the list only contains 1 entry

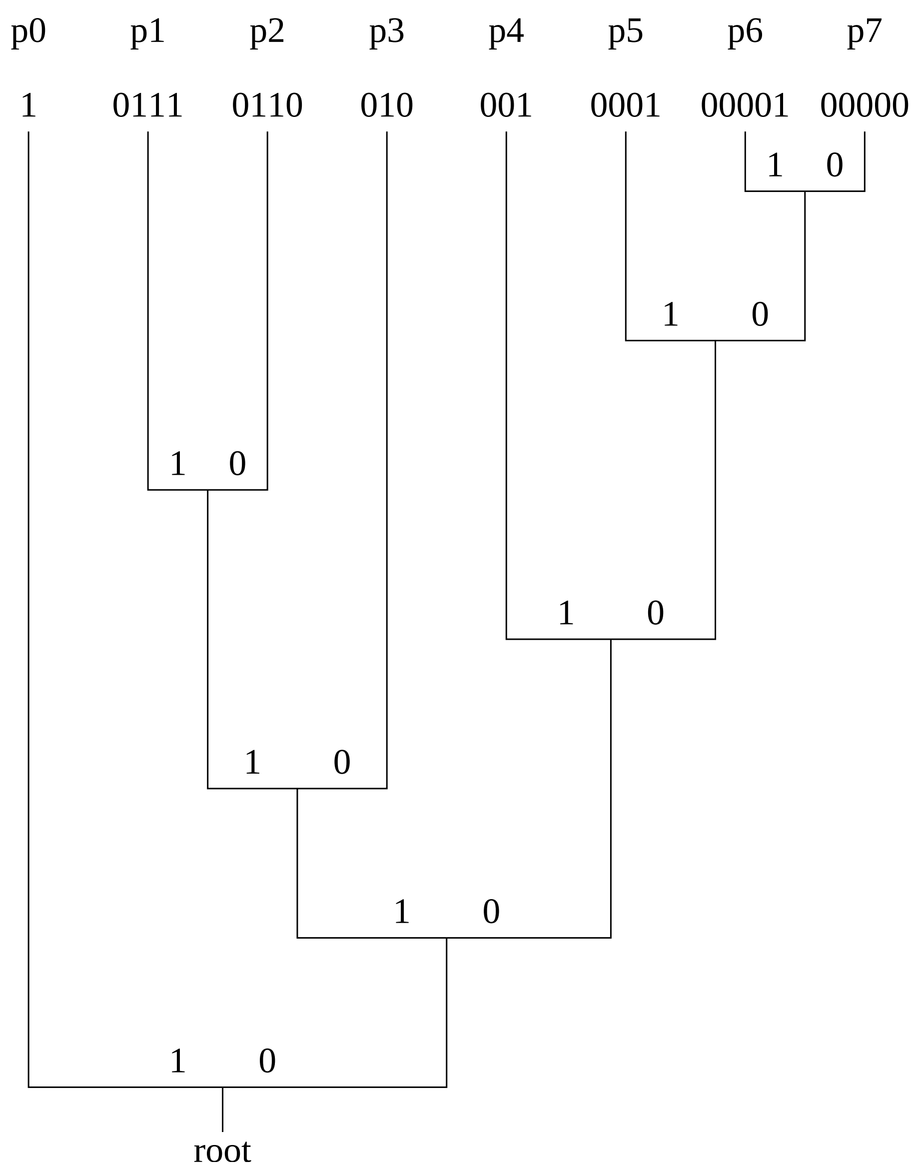
Recursive Merging for Huffman Code

- Example for $M = 8$ code



Resulting Huffman Code

- Binary codes given by path through tree



Upper Bound on Bit Rate of Huffman Code

- **Theorem:** For a Huffman code, $\bar{n}$ has the property that

$$H(X_n) \leq \bar{n} < H(X_n) + 1$$

- A Huffman code is within 1 bit of optimal efficiency
- Can we do better?

Coding in Blocks

- We can code blocks of symbols to achieve a bit rate that approaches the entropy of the source symbols.

$$\cdots, \underbrace{X_0, \cdots, X_{m-1}}_{Y_0}, \underbrace{X_m, \cdots, X_{2m-1}}_{Y_1}, \cdots$$

So we have that

$$Y_n = \left[X_{nm}, \cdots, X_{(n+1)m-1}\right]$$

where

$$Y_n \in \{0, \cdots, M^m - 1\}$$

Bit Rate Bounds for Coding in Blocks

- It is easily shown that $H(Y_n) = mH(X_n)$ and the number of bits per symbol X_n is given by $\bar{n}_x = \frac{\bar{n}_y}{m}$ where $\bar{n}_y$ is the number of bits per symbol for a Huffman code of Y_n.
- Then we have that

$$
\begin{aligned}
H(Y_n) \leq \; & \bar{n}_y \; < H(Y_n) + 1 \\
\frac{1}{m} H(Y_n) \leq \; & \frac{\bar{n}_y}{m} \; < \frac{1}{m} H(Y_n) + \frac{1}{m} \\
H(X_n) \leq \; & \frac{\bar{n}_y}{m} \; < H(X_n) + \frac{1}{m} \\
H(X_n) \leq \; & \bar{n}_x \; < H(X_n) + \frac{1}{m}
\end{aligned}
$$

- As the block size grows, we have

$$
\begin{aligned}
\lim_{m \to \infty} H(X_n) \; & \leq \; \lim_{m \to \infty} \bar{n}_x \leq H(X_n) + \lim_{m \to \infty} \frac{1}{m} \\
H(X_n) \; & \leq \; \lim_{m \to \infty} \bar{n}_x \leq H(X_n)
\end{aligned}
$$

- So we see that for a Huffman code of blocks with length m

$$
\lim_{m \to \infty} \bar{n}_x = H(X_n)
$$

Comments on Entropy Coding

- As the block size goes to infinity the bit rate approaches the entropy of the source

$$\lim_{m\to\infty} \bar{n}_x = H(X_n)$$

- A Huffman coder can achieve this performance, but it requires a large block size.
- As m becomes large M^m becomes very large $\Rightarrow$ large blocks are not practical.
- This assumes that X_n are i.i.d., but a similar result holds for stationary and ergodic sources.
- Arithmetic coders can be used to achieve this bitrate in practical situations.

Run Length Coding

- In some cases, long runs of symbols may occur. In this case, run length coding can be effective as a preprocessor to an entropy coder.
- Typical run length coder uses

  $$\cdots\ ,(\text{value}, \#\text{ of repetitions}), (\text{value}, \#\text{ of repetitions+1}), \cdots$$

  where 2^b is the maximum number of repetitions
- Example: Let $X_n \in \{0, 1, 2\}$

  $$\cdots | \underbrace{0000000}_{07} | \underbrace{111}_{13} | \underbrace{222222}_{26} | \cdots$$

- If more than 2^b repetitions occur, then the repetition is broken into segments

  $$\cdots | \underbrace{00000000}_{08} | \underbrace{00}_{02} | \underbrace{111}_{13} | \cdots$$

- Many other variations are possible.

Predictive Entropy Coder for Binary Images

- Uses in transmission of Fax images (CCITT G4 standard)
- Framework
  - Let X_s be a binary image on a rectangular lattice $s = (s_1, s_2) \in S$
  - Let W be a causal window in raster order
  - Determine a model for $p(x_s | x_{s+r}\, r \in W)$
- Algorithm

1. For each pixel in raster order

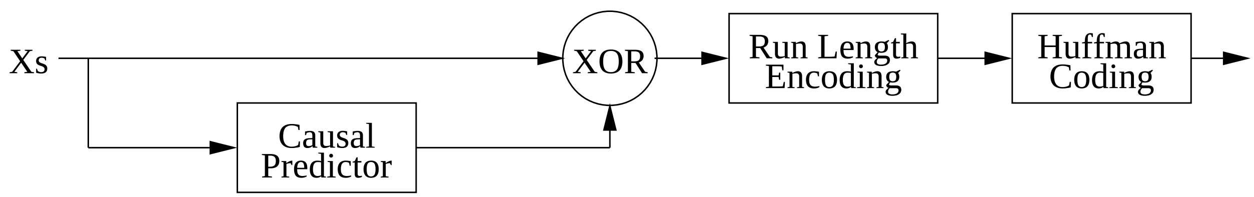
   (a) Predict

   $$\hat{X}_s = \begin{cases} 1 & \text{if } p(1|X_{s+r}\, r \in W) > p(0|X_{s+r}\, r \in W) \\ 0 & \text{otherwise} \end{cases}$$

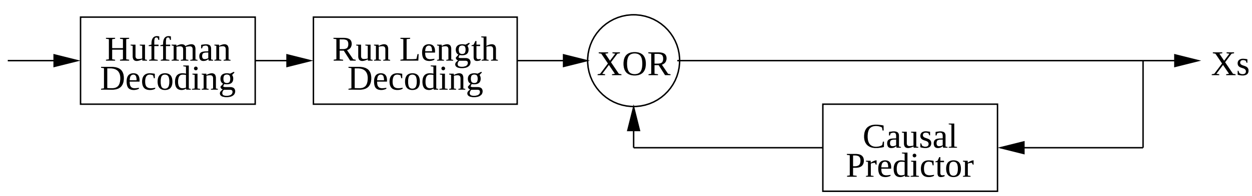
   (b) If $X_s = \hat{X}_s$ send 0; otherwise send 1
2. Run length code the result
3. Entropy code the result

Predictive Entropy Coder Flow Diagram

Encoder



Decoder



How to Choose $p(x_s|x_{s+r}\, r \in W)$?

- Non-adaptive method
  - Select typical set of training images
  - Design predictor based on training images
- Adaptive method
  - Allow predictor to adapt to images being coded
  - Design decoder so it adapts in same manner as encoder

Non-Adaptive Predictive Coder

- Method for estimating predictor
  1. Select typical set of training images
  2. For each pixel in each image, form $z_s = (x_{s+r_0}, \cdots, x_{s+r_{p-1}})$ where $\{r_0, \cdots, r_{p-1}\} \in W$.
  3. Index the values of z_s from $j = 0$ to $j = 2^p - 1$
  4. For each pixel in each image, compute

     $$h_s(i,j) = \delta(x_s = i)\delta(z_s = j)$$

     and the histogram

     $$h(i,j) = \sum_{s \in S} h_s(i,j)$$

  5. Estimate $p(x_s | x_{s+r}\, r \in W) = p(x_s | z_s)$ as

     $$\hat{p}(x_s = i | z_s = j) = \frac{h(i,j)}{\sum_{k=0}^{1} h(k,j)}$$

Adaptive Predictive Coder

- Adapt predictor at each pixel
- Update value of $h(i,j)$ at each pixel using equations

$$h(i,j) \leftarrow h(i,j) + \delta(x_s = i)\delta(z_s = j)$$

$$N(j) \leftarrow N(j) + 1$$

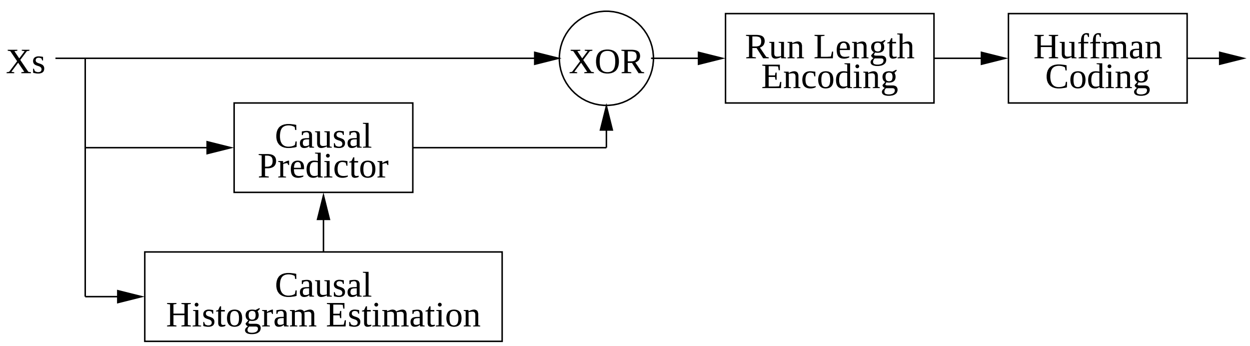
- Use updated values of $h(i,j)$ to compute new predictor at each pixel

$$\hat{p}(i|j) \leftarrow \frac{h(i,j)}{N(j)}$$

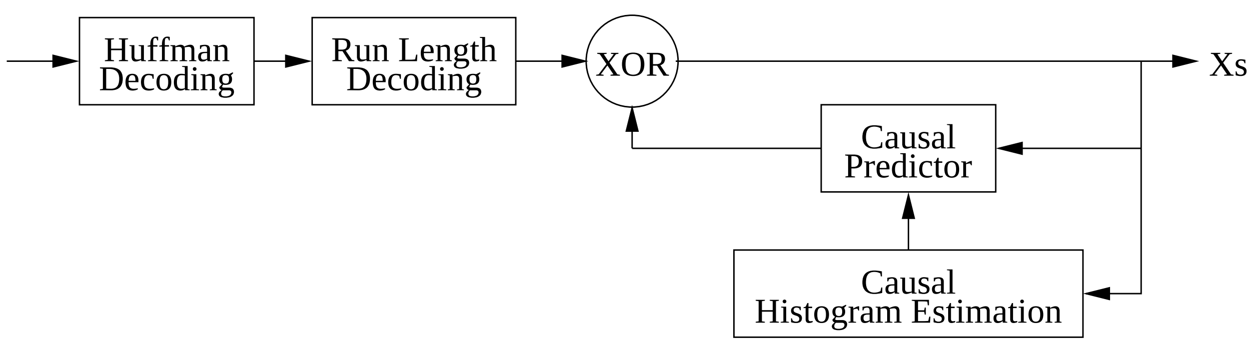
- Design decoder to track encoder

Adaptive Predictive Entropy Coder Flow Diagram

Encoder



Decoder



Lossy Source Coding

- Method for representing discrete-space signals with minimum distortion and bit-rate
- Outline
  - Rate-distortion theory
  - Karhunen-Loeve decorrelating Transform
  - Practical coder structures

Distortion

- Let X and Z be random vectors in $\mathbb{R}^M$. Intuitively, X is the original image/data and Z is the decoded image/data.

  Assume we use the squared error distortion measure given by

$$d(X,Y) = ||X - Z||^2$$

  Then the distortion is given by

$$D = E\left[d(X,Y)\right] = E\left[||X - Z||^2\right]$$

- This actually applies to any quadratic norm error distortion measure since we can define

$$\tilde{X} = AX \quad \text{and} \quad \tilde{Z} = AZ$$

  So

$$\tilde{D} = E\left[||\tilde{X} - \tilde{Z}||^2\right] = E\left[||X - Z||_B^2\right]$$

  where $B = A^t A$.

Lossy Source Coding: Theoretical Framework

- Notation for source coding

  $X_n \in \mathbb{R}^M$ for $0 \leq n < N$ - a sequence of i.i.d. random vectors

  $Y \in \{0,1\}^K$ - a K bit random binary vector.

  $Z_n \in \mathbb{R}^M$ for $0 \leq n < N$ - the decoded sequence of random vectors.

  $$X^{(N)} = (X_0, \cdots, X_{N-1})$$
  $$Z^{(N)} = (Z_0, \cdots, Z_{N-1})$$

  – Encoder function: $Y = Q(X_0, \cdots, X_{N-1})$
  – Decoder function: $(Z_0, \cdots, Z_{N-1}) = f(Y)$

- Resulting quantities

  Bit-rate $= \frac{K}{N}$

  $$\text{Distortion} = \frac{1}{N}\sum_{n=0}^{N-1} E\left[||X_n - Z_n||^2\right]$$

- How do we choose $Q(\cdot)$ to minimize the bit-rate and distortion?

Differential Entropy

- Notice that the information contained in a Gaussian random variable is infinite, so the conventional entropy $H(X)$ is not defined.

- Let X be a random vector taking values in $\mathbb{R}^M$ with density function $p(x)$. Then we define the differential entropy of X as

$$\begin{aligned} h(X) &= -\int_{x \in \mathbb{R}^M} p(x) \log_2 p(x) dx \\ &= -E\left[\log_2 p(X)\right] \end{aligned}$$

  $h(X)$ has units of bits

Conditional Entropy and Mutual Information

- Let X and Y be a random vectors taking values in $\mathbb{R}^M$ with density function $p(x, y)$ and conditional density $p(x|y)$.
- Then we define the differential conditional entropy of X given Y as

$$
\begin{aligned}
h(X|Y) &= -\int_{x\in\mathbb{R}^M}\int_{x\in\mathbb{R}^M} p(x,y)\log_2 p(x|y) \\
&= -E\left[\log_2 p(X|Y)\right]
\end{aligned}
$$

- The mutual information between X and Y is given by

$$
I(X;Y) = h(X) - h(X|Y) = I(Y;X)
$$

- **Important:** The mutual information is well defined for both continuous and discrete random variables, and it represents the reduction in uncertainty of X given Y.

The Rate-Distortion Function

- Define/Remember:
  - Let X_0 be the first element of the i.i.d. sequence.
  - Let $D \geq 0$ be the allowed distortion.
- For a specific distortion, D, the rate is given by

$$R(D) = \inf_Z \left\{ I(X_0; Z) : E\left[||X_0 - Z||^2 \right] \leq D \right\}$$

  where the infimum (i.e. minimum) over Z is taken over all random variables Z.

- Later, we will show that for a given distortion we can find a code that gets arbitrarily close to this optimum bit-rate.

Properties of the Rate-Distortion Function

- Properties of $R(D)$
  - $R(D)$ is a monotone decreasing function of D.
  - If $D \geq E\left[||X_0||^2\right]$, then $R(D) = 0$
  - $R(D)$ is a convex function of D

Shannon's Source-Coding Theorem

- Shannon's Source-Coding Theorem:
  For any $R' > R(D)$ and $D' > D$ there exists a sufficiently large N such that there is an encoder

$$Y = Q(X_0, \cdots, X_{N-1})$$

  which achieves

$$\text{Rate} = \frac{K}{N} \leq R'$$

  and

$$\text{Distortion} = \frac{1}{N} \sum_{n=0}^{N-1} E\left[||X_n - Z_n||^2 \right] \leq D'$$

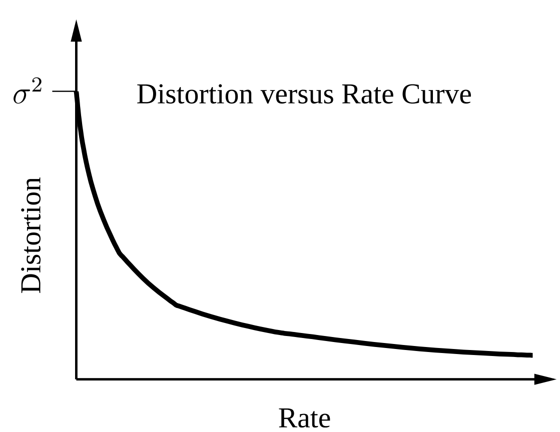
- Comments:
  - One can achieve a bit rate arbitrarily close to $R(D)$ at a distortion D.
  - Proof is constructive (but not practical), and uses codes that are randomly distributed in the space $\mathbb{R}^{MN}$ of source symbols.

Example 1: Coding of Gaussian Random Variables

- Let $X \sim N(0, \sigma^2)$ with distortion function $E\left[|X - Z|^2\right]$, then it can be shown that the rate-distortion function has the form

$$
\begin{aligned}
R(\delta) &= \max\left\{\frac{1}{2}\log\left(\frac{\sigma^2}{\delta}\right), 0\right\} \\
D(\delta) &= \min\left\{\sigma^2, \delta\right\}
\end{aligned}
$$

- Intuition:
  - δ is a parameter which represents the $\sqrt{\text{quantization step}}$
  - $\frac{1}{2}\log\left(\frac{\sigma^2}{\delta}\right)$ represents the number of bits required to encode the quantized scalar value.
  - Minimum number of bits must be ≥ 0.
  - Maximum distortion must be $\leq \sigma^2$.



Example 2: Coding of N Independent Gaussian Random Variables

- Let $X = [X_1, \cdots, X_{N-1}]^t$ with independent components such that $X_n \sim N(0, \sigma_n^2)$, and define the distortion function to be

$$\text{Distortion} = E\left[||X - Z||^2 \right]$$

Then it can be shown that the rate-distortion function has the form

$$\begin{aligned} R(\delta) &= \sum_{n=0}^{N-1} \max\left\{ \frac{1}{2} \log\left(\frac{\sigma_n^2}{\delta} \right), 0 \right\} \\ D(\delta) &= \sum_{n=0}^{N-1} \min\left\{ \sigma_n^2, \delta \right\} \end{aligned}$$

- Intuition:
  - In an optimal coder, the quantization step should be approximately equal for each random variable being quantized.
  - The bit rate and distortion both add for each component.
  - It can be proved that this solution is optimal.

Example 3: Coding of Gaussian Random Vector

- Let $X \sim N(0, R)$ be a N dimensional Gaussian random vector, and define the distortion function to be

$$\text{Distortion} = E\left[||X - Z||^2\right]$$

- Analysis: We know that we can always represent the covariance in the form

$$R = T^t \Lambda T$$

where the columns of T are the eigenvectors of R, and $\Lambda = diag\{\sigma_0^2, \cdots, \sigma_{N-1}^2\}$ is a diagonal matrix of eigenvalues. We can then decorrelate the Gaussian random vector with the following transformation.

$$\tilde{X} = T^t X$$

From this we can see that $\tilde{X}$ has the covariance matrix given by

$$\begin{aligned} E\left[\tilde{X}\tilde{X}^t\right] &= E\left[T^t X X^t T\right] \\ &= T^t E\left[X X^t\right] T = T^t R T = \Lambda \end{aligned}$$

So therefore, $\tilde{X}$ meets the conditions of Example 2. Also, we see that

$$E\left[||X - Z||^2\right] = E\left[||\tilde{X} - \tilde{Z}||^2\right]$$

where $\tilde{X} = T^t X$ and $\tilde{Z} = T^t Z$ because T is an orthonormal transform.

Example 3: Coding of Gaussian Random Vector (Result)

- Let $X \sim N(0, R)$ be a N dimensional Gaussian random vector, and define the distortion function to be

$$\text{Distortion} = E\left[||X - Z||^2\right]$$

Then it can be shown that the rate-distortion function has the form

$$R(\delta) = \sum_{n=0}^{N-1} \max\left\{\frac{1}{2}\log\left(\frac{\sigma_n^2}{\delta}\right), 0\right\}$$

$$D(\delta) = \sum_{n=0}^{N-1} \min\left\{\sigma_n^2, \delta\right\}$$

where $\sigma_0^2, \cdots, \sigma_{N-1}^2$ are the eigenvalues of R.

- Intuition:
  - An optimal code requires that the components of a vector be decorrelated before source coding.

Example 4: Coding of Stationary Gaussian Random Process

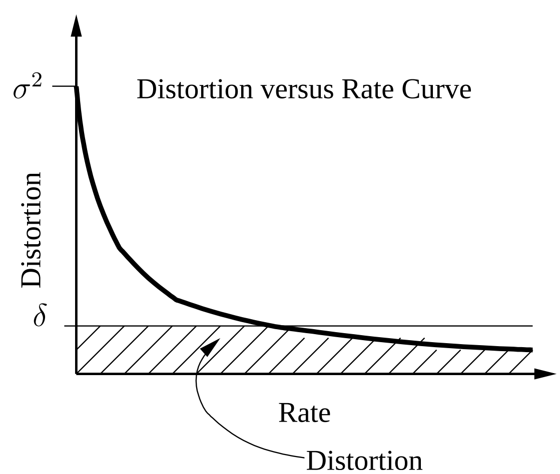
- Let X_n be a stationary Gaussian random process with power spectrum $S_x(\omega)$, and define the distortion function to be

$$\text{Distortion} = E\left[|X_n - Z|^2\right]$$

Then it can be shown that the rate-distortion function has the form

$$\begin{aligned} R(\delta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \max\left\{\frac{1}{2}\log\left(\frac{S_x(\omega)}{\delta}\right), 0\right\} d\omega \\ D(\delta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \min\left\{S_x(\omega), \delta\right\} d\omega \end{aligned}$$

- Intuition:
  - The Fourier transform decorrelates a stationary Gaussian random process.
  - Frequencies with amplitude below δ are clipped to zero.



The Discrete Cosine Transform (DCT)

- DCT (There is more than one version)

$$F(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, c(k) \cos\left(\frac{\pi(2n+1)k}{2N} \right)$$

where

$$c(k) = \left\{ \begin{array}{ll} 1 & k = 0 \\ \sqrt{2} & k = 1, \cdots, N-1 \end{array} \right\}$$

- Inverse DCT (IDCT)

$$f(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F(k)\, c(k) \cos\left(\frac{\pi(2n+1)k}{2N} \right)$$

- Comments:
  - In this form, the DCT is an orthonormal transform. So if we define the matrix F such that

    $$F_{n,k} = \ c(k) \cos\left(\frac{\pi(2n+1)k}{2N} \right) \ ,$$

    then $F^{-1} = F^H$ where

    $$F^{-1} = \left[F^t \right]^* = F^H$$

  - Takes and N-point real valued signal to an N-point real valued signal.

Relationship Between DCT and DFT

- Let us define the padded version of $f(n)$ as

$$f_p(n) = \begin{cases} f(n) & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq 2N-1 \end{cases}$$

and its $2N$-point DFT denoted by $F_p(k)$. Then the DCT can be written as

$$\begin{aligned}
F(k) &= \frac{c(k)}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) \cos\left(\frac{\pi(2n+1)k}{2N}\right) \\
&= \frac{c(k)}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) Re\left\{ e^{-j\frac{2\pi(2n+1)k}{2N}} e^{-j\frac{\pi k}{2N}} \right\} \\
&= \frac{c(k)}{\sqrt{N}} Re\left\{ e^{-j\frac{\pi k}{2N}} \sum_{n=0}^{2N-1} f_p(n) e^{-j\frac{2\pi(2n+1)k}{2N}} \right\} \\
&= \frac{c(k)}{\sqrt{N}} Re\left\{ e^{-j\frac{\pi k}{2N}} F_p(k) \right\} \\
&= \frac{c(k)}{\sqrt{N}} \left(F_p(k) e^{-j\frac{\pi k}{2N}} + F_p(k) e^{+j\frac{\pi k}{2N}} \right) \\
&= \frac{c(k) e^{-j\frac{\pi k}{2N}}}{\sqrt{N}} \left(F_p(k) + \left(F_p(k) e^{-j\frac{2\pi k}{2N}} \right)^* \right)
\end{aligned}$$

Interpretation of DFT Terms

- Consider the inverse DCT for each of the two terms

$$F_p(k) \Leftrightarrow f_p(n)$$

- $DCT\{f_p(n)\} \Rightarrow F_p(k)$
- $DCT\{f_p(-n+N-1)\} \Rightarrow \left(F_p(k)e^{-j\frac{2\pi k}{2N}}\right)^*$
- Simple example for $N = 4$

$$\begin{aligned} & f(n) \\ & = [f(0), f(1), f(2), f(3)] \end{aligned}$$

$$\begin{aligned} & f_p(n) \\ & = [f(0), f(1), f(2), f(3), 0, 0, 0, 0] \end{aligned}$$

$$\begin{aligned} & f_p(-n+N-1) \\ & = [0, 0, 0, 0, f(3), f(2), f(1), f(0)] \end{aligned}$$

$$\begin{aligned} & f(n) + f_p(-n+N-1) \\ & = [f(0), f(1), f(2), f(3), f(3), f(2), f(1), f(0)] \end{aligned}$$

Relationship Between DCT and DFT (Continued)

- So the DCT is formed by $2N$-point DFT of $f(n)+f(-n+N-1)$.

$$F(k) = \frac{c(k)e^{-j\frac{\pi k}{2N}}}{\sqrt{N}} DFT_{2N}\left\{f(n) + f(-n+N-1)\right\}$$