Filtered Random Processes

• Consider the 2-D linear system

$$Y(m,n) = h(m,n) * X(m,n)$$

where X(m,n) is a 2-D wide sense stationary random process.

• It may be easily shown that

$$R_y(m,n) = h(m,n) * h(-m,-n) * R_x(m,n)$$

$$S_y(e^{j\mu}, e^{j\nu}) = |H(e^{j\mu}, e^{j\nu})|^2 S_x(e^{j\mu}, e^{j\nu})$$

2D Gaussian White Noise

- Definition:
 - X(m,n) independent identically distributed (i.i.d.) Gaussian random variables with distribution N(0,1).
- Then
 - -X(m,n) is wide sense stationary with

$$\mu(m, n) = E[X(m, n)] = 0$$

$$R_x(k, l) = E[X(0, 0)X(k, l)] = \delta(k, l)$$

$$S_x(e^{j\mu}, e^{j\nu}) = DSFT \{R_x(k, l)\}$$
= 1

Filtered White Noise

- Definitions:
 - -X(m,n) independent identically distributed (i.i.d.) Gaussian random variables with distribution N(0,1).
 - -Y(m,n) = h(m,n) * X(m,n).
- Then
 - -Y(m,n) is wide sense stationary with

$$S_{y}(e^{j\mu}, e^{j\nu}) = |H(e^{j\mu}, e^{j\nu})|^{2} S_{x}(e^{j\mu}, e^{j\nu})$$

$$= |H(e^{j\mu}, e^{j\nu})|^{2} \cdot 1$$

$$R_{y}(k, l) = h(m, n) * h(-m, -n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m, n)h(m + k, n + l)$$

- $R_y(k, l)$ is the autocorrelation of h(m, n) with itself.

Causal Prediction

- Y_s is a 2-D wide sense stationary zero mean Gaussian random process.
- Define
 - The past values are $Y_{\leq s} = \{Y_r : r \leq s\}$.
 - The minimum mean squared error (MMSE) predictor of Y_s given the past is

$$\hat{Y}_s = E[Y_s | Y_{< s}]$$

– The prediction error is $X_s = Y_s - \hat{Y}_s$.

Properties of Causal Predictors

• Fact 1: (WLOG, assume r < s.)

$$E[X_{s}X_{r}] = E[E[X_{s}X_{r}|Y_{< s}]]$$

$$= E[E[(Y_{s} - \hat{Y}_{s})(Y_{r} - \hat{Y}_{r})|Y_{< s}]]$$

$$= E[E[(Y_{s} - \hat{Y}_{s})|Y_{< s}](Y_{r} - \hat{Y}_{r})]$$

$$= E[(E[Y_{s}|Y_{< s}] - \hat{Y}_{s})(Y_{r} - \hat{Y}_{r})]$$

$$= E[(\hat{Y}_{s} - \hat{Y}_{s})(Y_{r} - \hat{Y}_{r})]$$

$$= E[0(Y_{r} - \hat{Y}_{r})] = 0$$

- Fact 2: $\sigma^2 \stackrel{\triangle}{=} E[X_s^2]$ is the prediction variance.
- Fact 3: The predictor must be space invariant and linear.

$$\hat{Y}_s = \sum_{r > (0,0)} h_r Y_{s-r}$$

Autoregressive (AR) Processes

- Definitions:
 - Y_s 2-D wide sense stationary zero mean Gaussian random process.
 - h_s MMSE linear predictor for Y_s .
 - $X_s = Y_s h_s * Y_s$ predictor error.
- If h_s is FIR, then Y_s is known as an autoregressive (AR) process.

Properites of AR Processes

• Remember that

$$X_s = Y_s - h_s * Y_s$$

- Then
 - We know that

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Since X_s is white noise,

$$R_x(s) = \sigma^2 \delta(s)$$
$$S_x(e^{j\mu}, e^{j\nu}) = \sigma^2$$

– So the power spectrum of Y_s is given by

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\sigma^2}{|1 - H(e^{j\mu}, e^{j\nu})|^2}$$

Spectral Estimate Using AR Processes

- Compute MMSE linear predictor \hat{h}_s for Y_s .
- Compute the prediction variance

$$\hat{\sigma}^2 = \frac{1}{|S|} \sum_{s \in S} |Y_s - h_s * Y_s|^2$$

where S is a finite set of points in plain, and |S| is the number of points in S.

• Estimate the power spectrum

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\hat{\sigma}^2}{\left|1 - \hat{H}(e^{j\mu}, e^{j\nu})\right|^2}$$

• Can produce a more accurate estimate of the power spectrum.

Generating AR Processes

- Select a causal prediction filter h_s .
- ullet Apply IIR filter to white noise random process X_s

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- \bullet Y_s is sometimes referred to as a white noise driven process.
- Do linear FIR prediction filters \hat{h}_s always form a stable IIR filter?
 - In 1-D, yes.
 - In 2-D, not always!
- Other problems:
 - Causal ordering of points may cause asymmetric artifacts in results.
 - Complexity increases rapidly with IIR filter order P.