EE 637 Study Solutions - Assignment 4

Topics: Random processes, spectral estimation, and eigen-image analysis

Spring 2010 Final: Problem 5 (power spectrum and MMSE prediction)

Let $X = [x_1, \cdots, x_N]$ be a $P \times N$ matrix formed by P dimensional column vectors, $x_n \in \Re^P$ where $N < P$. We will assume that each column vector is an independent multivariate Gaussian random vector with distribution $N(0, R)$, where R is a positive definite and symmetric matrix.

Furthermore, let $X = U\Sigma V^t$ be the singular value decomposition of X, where $\Sigma_{i,i} \geq \Sigma_{j,j}$ when $i < j$.

a) Write a simple matrix expression for the sample covariance, $\hat{R}$.

Solution:

$$\hat{R} = \frac{1}{N} X X^t$$

b) Let $\hat{R} = E\Lambda E^t$ be the eigen decomposition of the sample covariance matrix, where E is the orthonormal transform with eigenvectors as columns and Λ is the diagonal matrix of eigen values. (Without loss of generality assume that the eigenvalues are ordered from largest to smallest so that $\Lambda_{i,i} \geq \Lambda_{j,j}$ when $i < j$.

How many non-zero eigenvalues does the matrix $\hat{R}$ contain?

Solution:

There are N non-zero eigenvalues.

c) Specify the eigenvectors and eigenvalues of $\hat{R}$ in terms of the SVD of X.

Solution:

$$\begin{aligned}
\hat{R} &= \frac{1}{N} X X^t \\
&= \frac{1}{N} U\Sigma V^t \left(U\Sigma V^t\right)^t \\
&= \frac{1}{N} U\Sigma V^t V \Sigma U^t \\
&= \frac{1}{N} U\Sigma^2 U^t
\end{aligned}$$

$$\text{So, } E = U \text{ and } \Lambda = \frac{1}{N}\Sigma^2$$

d) In some application, you can only use two numbers to specify each vector, x_n. So each vector must be approximated by

$$x_n \approx a_n e_1 + b_n e_2$$

where $e_1 \in \Re^P$ and $e_2 \in \Re^P$ are two orthonormal vectors, and a_n and b_n are two scalar values used to specify each vector, x_n.

What is the best choice of e_1 and e_2?

Solution:

The vectors e_1 and e_2 should be the first and second columns of U, so that e_1 is the singular vector corresponding to the largest singular value Σ_{11}, and e_2 is the singular vector corresponding to the second largest singular value Σ_{22}.

Spring 2010 Exam 1: Problem 3 (power spectrum and MMSE prediction)

Let y_n be a wide-sense stationary, jointly Gaussian, zero-mean, discrete-time random process. Then we know that the minimum mean squared error (MSEE) predictor has the form

$$\begin{aligned} \hat{y}_n &= E\left[y_n | \, y_k \text{ for } k < n\right] \\ &= \sum_{i=1}^{\infty} h_{n,i} y_{n-i} \end{aligned}$$

for some scalar constants $h_{n,i}$.

a) How are the functions $h_{n,i}$ and $h_{k,i}$ related for all n, k, and i? Provide a precise justification for your answer.

Solution:

$h_{n,i} = h_{k,i}$, because Y_n is a stationary random process.

b) Consider the function $z_n = y_{-n}$. What is the MMSE predictor for z_n? Provide a precise justification for your answer.

Solution:

Notice that:

$$\begin{aligned} R_z(k) &= E\left[Z_n Z_{n+k}\right] \\ &= E\left[Y_{-n} Y_{-n-k}\right] \\ &= E\left[Y_{m+k} Y_m\right], \text{ using } m = -n-k \\ &= E\left[Y_m Y_{m+k}\right] \\ &= R_y(k) \end{aligned}$$

$\Rightarrow Y_n$ and Z_n have the same distribution.

$\Rightarrow Y_n$ and Z_n have the same predictor.

$$\hat{z_n} = \sum_{i=1}^{\infty} h_i z_{n-i}$$

c) What is the autocorrelation of $x_n = y_n - \hat{y}_n$. Provide a precise justification for your answer.

Solution:

$$R_X(k) = \delta_k \sigma_x^2 \text{ (see proof in notes)}$$

d) Derive an expression for the power spectrum of the random process, y_n.

Solution:

$$S_x\left(e^{j\omega}\right) = \frac{\sigma_x^2}{\left|1 - H\left(e^{j\omega}\right)\right|^2}$$

Spring 2010 Exam 1: Problem 1 (power spectrum and IIR filters)

Consider the following 2-D discrete-time linear system.

$$y(m,n) = x(m,n) + ay(m-1,n) + by(m,n-1) - aby(m-1,n-1)$$

where a and b are scalar constants.

a) Compute the transfer function $H(z_1, z_2)$ for the system.

Solution:

$$Y(z_1,z_2)\left(1 - az_1^{-1} - bz_2^{-1} + abz_1^{-1}z_2^{-1}\right) = X(z_1,z_2)$$
$$H(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)} = \frac{1}{\left(1-az_1^{-1}\right)\left(1-bz_2^{-1}\right)}$$

b) Compute the impulse response $h(m,n)$ for the system.

Solution:

$$h(m,n) = (a^m u(m))\,(b^n u(n))$$

c) For what values of a and b is the system stable.

Solution:

$$|a| < 1, \text{ and } |b| < 1$$

d) Compute $S_y(e^{j\mu}, e^{j\nu})$, the power spectrum of $y(m,n)$, when $x(m,n)$ is a set of i.i.d. $\mathcal{N}(0,\sigma^2)$ random variables.

Solution:

$$S_x\left(e^{j\mu}, e^{j\nu}\right) = \sigma^2$$
$$S_y\left(e^{j\mu}, e^{j\nu}\right) = \sigma^2 \left|H\left(e^{j\mu}, e^{j\nu}\right)\right|^2$$
$$= \frac{\sigma^2}{|1-ae^{-j\mu}|^2\,|1-be^{-j\nu}|^2}$$