EE 637 Final

May 1, Spring 2019

Name: ____

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$
$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$
$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$
$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$
$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$
$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$
$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$
$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$
$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy$$
$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$
$$(n+1) a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{(1 - a e^{-j\omega})^2}$$

- Rep and Comb relations

$$\text{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$
$$\text{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$
$$\text{comb}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}} [X(f)]$$
$$\text{rep}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$
$$Y(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T} \right)$$
$$s(t) = \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT)$$
$$S(e^{j\omega}) = Y\left(e^{j\omega T}\right)$$

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Problem 1.(20pt)

Consider the following 2D FIR filter with input $x(m,n)$ and output $y(m,n)$

$$y(m,n) = x(m,n) + \lambda\left(\delta(m,n) - w(m,n)\right) * x(m,n) \ , \tag{1}$$

where $*$ denotes 2D convolution and the function $w(m,n)$ is given by

$$w(m,n) = \begin{cases} \frac{1}{9} & \text{for } |m| \leq 1 \text{ and } |n| \leq 1 \\ 0 & \text{otherwise} \end{cases} .$$

a) Calculate a closed form expression for $W(e^{j\mu}, e^{j\nu})$, the DSFT of $w(m,n)$.

b) Calculate a closed form expression for the impulse response, $h(m,n)$, of the 2D filter of equation (1).

c) Calculate a closed form expression for the transfer function, $H(e^{j\mu}, e^{j\nu})$, of the 2D filter of equation (1).

d) Sketch the function, $H(e^{j\mu}, e^{j0})$, for $\lambda = \frac{1}{2}$ and $\mu \in [-\pi, \pi]$.

e) What is the purpose of the 2D filter in equation (1)?

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Problem 2.(20pt)

Consider an MRI that only images in one dimension, x. So for example, the object being imaged might be a thin rod oriented along the x-dimension. In this example, assume that the magnetic field strength at each location is given by

$$M(x,t) = M_o + G(t)x$$

where M_o is the static magnetic field strength and $G(t)x$ is the linear gradient field in the x dimension. Furthermore, let γ denote the gyromagnetic constant for hydrogen, and let $a(x)$ denote the quantity of hydrogen per unit distance along the length of the rod.

a) Calculate $\omega(x,t)$, the frequency of precession of a hydrogen atom at location x and time t.

b) Calculate $\phi(x,t)$, the phase of precession of a hydrogen atom at location x and time t assuming that $\phi(x,0) = 0$.

c) Calculate $r(x,t)$, the signal radiated from hydrogen atoms in the interval $[x, x+dx]$ at time t.

d) Calculate $r(t)$, the total signal radiated from hydrogen atoms along the entire object.

e) Describe in words how one can recover the quantity $a(x)$ from the measurements $r(t)$.

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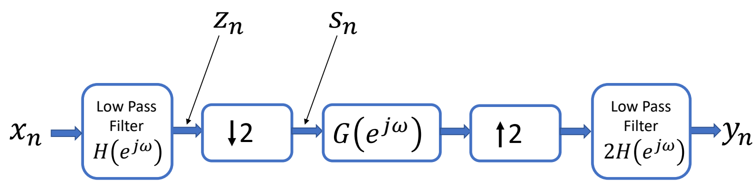
Problem 3.(20pt)



Consider the multi-rate system shown in the diagram with input x_n and output y_n. The first low pass filter is given by $H(e^{j\omega})$, and the second low pass filter is given by $2H(e^{j\omega})$. In between, there is a discrete-time filter,

$$G(e^{j\omega}) = \Lambda(\omega/\pi) \text{ for } |\omega| < \pi$$

a) Specify the frequency response, $H(e^{j\omega})$, required for the first low pass filter so that there is no aliasing in the down sampling. (Use the simplest form of this filter as was done in class.)

b) Find the impulse response, h_n, corresponding to this low pass filter.

c) Sketch the function $G(e^{j\omega})$ on the interval $[-2\pi, 2\pi]$.

d) Assuming that $x_n = \delta_n$, sketch $|Z(e^{j\omega})|$, the magnitude of the frequency response for z_n.

e) Assuming that $x_n = \delta_n$, sketch $|S(e^{j\omega})|$, the magnitude of the frequency response for s_n.

f) Assuming that $x_n = \delta_n$, sketch the magnitude of the frequency response for y_n, i.e., $Y(e^{j\omega})$.

g) Is this entire system linear and time invariant, i.e., LTI? Justify your answer.

h) Specify the frequency response of the entire system.

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Problem 4.(20pt)

Let $X(m,n)$ be a 2D discrete-space random process formed by independent identically distributed (i.i.d.) Gaussian random variables with mean 0 and variance 1 where m corresponds to the horizontal index (left-to-right) and n corresponds to the vertical index (top-to-bottom). Furthermore, let

$$Y(m,n) = X(m,n) + 0.8Y(m-1,n) + 0.9Y(m,n-1) - 0.72Y(m-1,n-1) \ .$$

a) Calculate the autocorrelation, $R_X(m,n)$, for the random process $X(m,n)$.

b) Calculate the power spectrum, $S_X(e^{j\mu}, e^{j\nu})$, for the random process $X(m,n)$.

c) Calculate frequency response, $H(e^{j\mu}, e^{j\nu})$, for the LSI system with input $X(m,n)$ and output $Y(m,n)$. (Put this in its most simplified form.)

d) Calculate the power spectrum, $S_Y(e^{j\mu}, e^{j\nu})$, for the random process $Y(m,n)$.

e) Sketch the 2D impulse response $h(m,n)$. (You can sketch the 2D discrete-space function in any descriptive way you prefer, but one possibility is to use an approximate contour plot.)

f) Sketch the 2D power spectrum $S_Y(e^{j\mu}, e^{j\nu})$ on the region $[-\pi,\pi] \times [-\pi,\pi]$. (You can sketch the 2D function in any descriptive way you prefer, but one possibility is to use an approximate contour plot.)

g) Calculate the MMSE predictor $\hat{Y}(m,n) = E\left[Y(m,n) | Y(l,k) \text{ for } l < m \text{ and } k < n\right]$.

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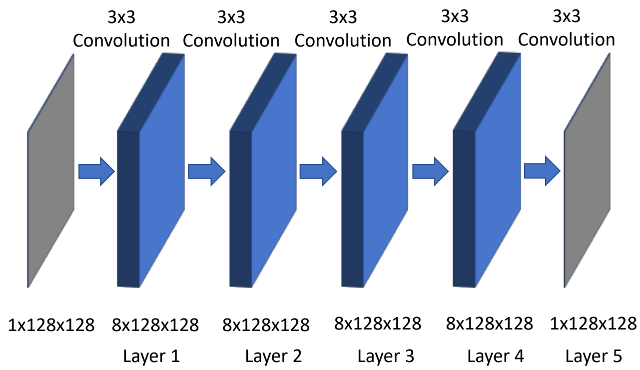
Problem 5.(20pt)



Consider a convolutional neural network (CNN) designed to denoise images as shown in the figure above. The input to the CNN is a noisy image Y and the output is $\hat{X}$, an estimate of the noiseless image X. More specifically,

$$Y = X + W$$

where W is i.i.d. Gaussian $N(0, \sigma^2)$ noise for $\sigma = 0.2$, and X is a 128×128 image from a large database of natural images with pixels in the range $[0, 1]$. Each layer of the CNN uses a 3×3 convolution kernel followed by a ReLU nonlinearity.

The loss function is given by

$$\text{Loss} = \frac{1}{128^2 N} \sum_{k=1}^{N} \|\hat{X}_k - X_k\|^2$$

where N is the number of ground truth images used. The CNN is trained on 100 images and then tested on 100 different images. In both cases, the Loss function is evaluated on the set of images to yield $\text{Loss}_{train} = (0.01)^2$ and $\text{Loss}_{test} = (0.15)^2$.

a) How many parameters are required for Layer 1? (Hint: Include the offset and convolution parameters for each layer.)

b) How many parameters are required for Layer 2?

c) How many parameters are required for the entire CNN?

d) Why is Loss_{test} larger than Loss_{train}?

e) What can be done the improve the results for this CNN? Justify your answer.