EE 637 Midterm II

April 6, Spring 2018

Name: (1 pt) KEY

Instructions:

- This is a 50 minute exam containing **3** problems with a total of 100 points.
- You may **only** use your brain and a pencil (or pen) and the included "Fact Sheet" to complete this exam.
- You **may not** use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You **may not** communicate with any person other than the official proctor during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\mathrm{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(f) &= Y\left(e^{j2\pi f T}\right) \end{aligned}$$

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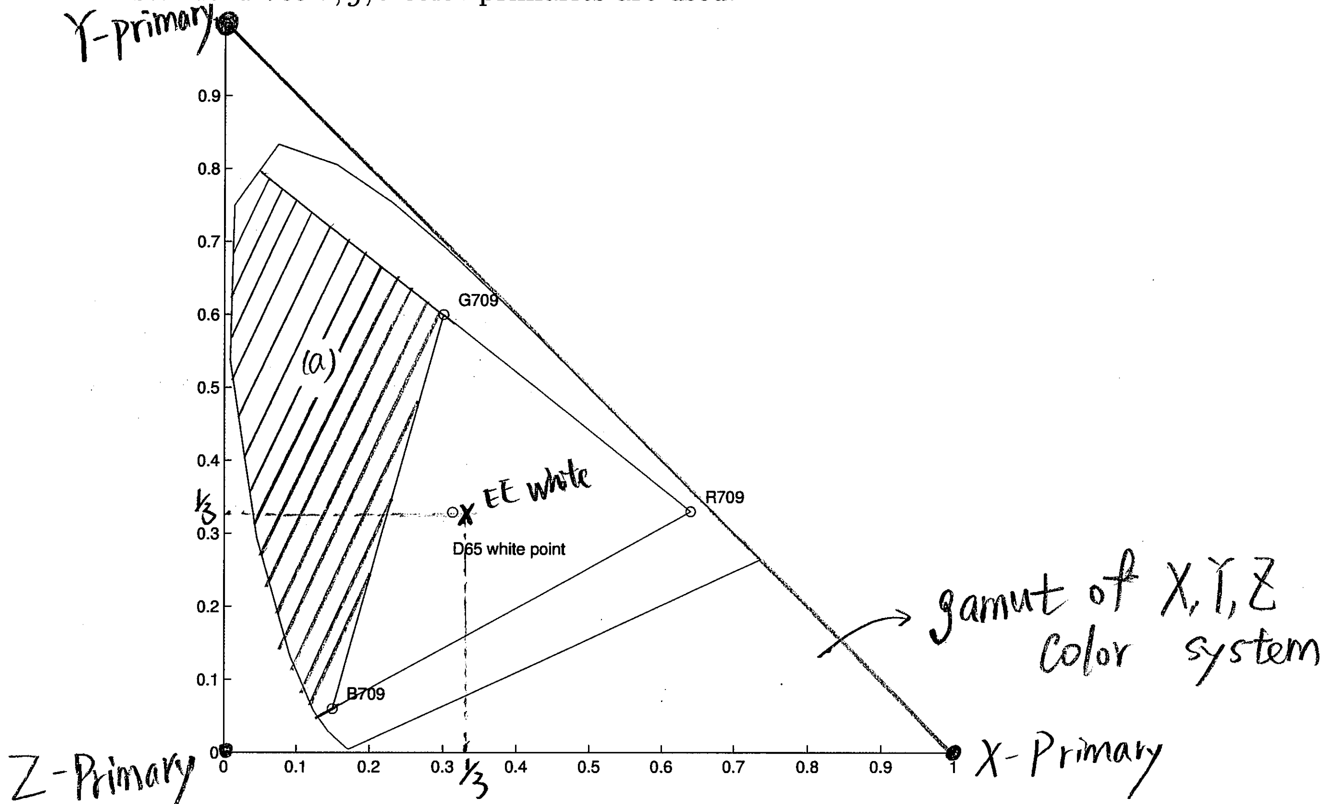
Problem 1.(33pt) Consider the standard chromaticity diagram below, and for all questions assume that standard 709 r, g, b color primaries are used.



a) Draw the region on the diagram corresponding to $r < 0$, $g > 0$, $b > 0$.

b) Draw a point corresponding to the color primaries for X, Y, Z, and label the three points "X-primary", "Y-primary", and "Z-primary".

c) Draw a triangle corresponding to the gamut of the X, Y, Z color system, when the three tristimulus values are assumed positive.

d) Do all positive values of X, Y, Z correspond to real colors? Why or why not?

e) Draw a point corresponding to equal energy white. How would it look different than D65 white point?

d) No, only colors inside "horse hue" are real. Colors outside that but within gamut are imaginary.

e) EE white $(\frac{1}{3}, \frac{1}{3})$ is more reddish than D65

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Problem 2.(33pt)

Let X be a $n \times 1$ vector of zero mean i.i.d. Gaussian $N(0,1)$ random variables, and let

$$\hat{X} = Z\theta$$

be an estimate of X where

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

is an $n \times p$ matrix with independent multivariate Gaussian rows distributed as $N(0, R)$ where

$$R = E\left[z_k^t z_k\right]$$

for any row z_k. Also, assume that X and Z are jointly Gaussian.

Furthermore, define the $1 \times n$ row vector $b = E\left[X_k z_k\right]$, and the scalar $a = E\left[|X_k|^2\right]$.

a) Derive an expression for the mean squared error (MSE) as a function of R, b, and θ.

b) Derive an expression for the vector, $\hat{\theta}$ that achieves the minimum mean squared error (MMSE). (Hint: Set the gradient to zero.)

c) Give specific formulas for practical estimates of R, b, and a denoted by $\hat{R}$, $\hat{b}$, and $\hat{a}$.

d) For each of the following quantities, state whether they are random or non-random: R, b, a, $\hat{R}$, $\hat{b}$, and $\hat{a}$.

(a) $MSE = \mathbb{E}[\|X - \hat{X}\|^2] = \mathbb{E}[X^T X] - 2\,\mathbb{E}[X^T Z]\theta + \theta^T \mathbb{E}[Z^T Z]\theta$

$= na - 2nb\theta + n\,\theta^T R \theta$

(b) $\dfrac{\partial MSE}{\partial \theta} = -2nb + 2n\,\theta^T R = 0$

$\hat{\theta}^T R = b$

$\hat{\theta} = R^{-1} b^T$

(c) $\hat{R} = \frac{1}{n} Z^T Z = \frac{1}{n} \sum_{i=1}^{n} Z_i^T Z_i$

$\hat{b} = \frac{1}{n} X^T Z = \frac{1}{n} \sum_{i=1}^{n} X_i Z_i$

$\hat{a} = \frac{1}{n} X^T X = \frac{1}{n} \sum_{i=1}^{n} X_i^2$

(d) $\hat{R}, \hat{b}, \hat{a}$ random

R, b, a not random

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Problem 3.(33pt)

Consider a person watching a 65" (diagonal dimension) 4K TV at a viewing distance of $d = 6ft = 72"$. The 4K TV displays a pixel array of size 3840×2160 with an aspect ratio of 16×9.

a) Give an expression for the width of the display. (Leave as a fraction since you don't have a calculator.)

b) Give an expression for the number of pixels per inch. (Leave as a fraction since you don't have a calculator.)

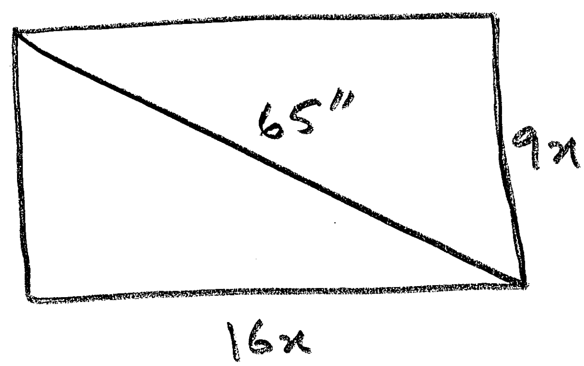
c) Give an expression for the number of pixels per degree. (Leave as a fraction since you don't have a calculator.)

d) Calculate a reasonable approximation to the value of answer c) without a calculator.

e) Would you expect a person to be able to see a grid of pixels with alternating intensities of $(r, g, b) = (0, 0, 0)$ and $(r, g, b) = (255, 255, 255)$ at the given distance of $6ft$?

(a) $(16x)^2 + (9x)^2 = 65^2$

$\Rightarrow x = \dfrac{65}{\sqrt{337}}$



width $= \dfrac{16 \times 65}{\sqrt{337}}$

height $= \dfrac{9 \times 65}{\sqrt{337}}$

(b) $PPI = \dfrac{3840}{\text{width}} = \dfrac{3840 \times \sqrt{337}}{16 \times 65}$

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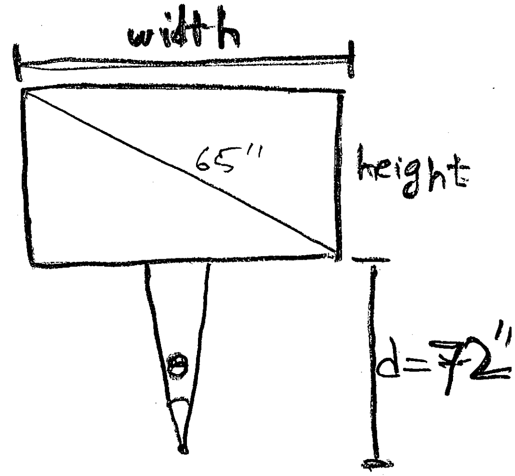

(c) $$PPD = PPI \times \frac{\text{inches}}{\text{degree}}$$

$$\approx PPI \times \frac{d\pi}{180} \quad \because \text{ for small } \theta$$

$$= \frac{3840 \times \sqrt{337}}{16 \times 65} \times \frac{72\pi}{180}$$

$$\approx 80$$

(d) $$\tilde{f}_0 = PPD \times \frac{\text{cycles}}{\text{pixel}}$$ (CPD = cycles per degree)

$$= 80 \times \frac{1}{2} = 40.$$

High frequency cut off of human contrast sensitivity function is around 10 cpd

So we don't expect a person to see the grid of pixels

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