EE 637 Final

April 30, Spring 2018

Name: ____

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\mathrm{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(f) &= Y\left(e^{j2\pi f T}\right) \end{aligned}$$

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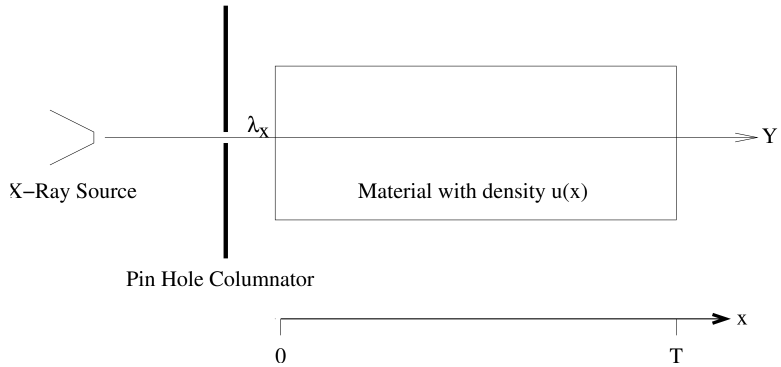
Problem 1.(20pt)

Consider an X-ray imaging system shown in the figure below.



Photons are emitted from an X-ray source and collimated by a pin hole in a lead shield. The collimated X-rays then pass in a straight line through an object of length T with density $u(x)$ where x is the depth into the object measured in units of cm, and $\mu(x)$ is measured in units of cm^{-1}. The expected number of photons at a depth x is given by

$$\lambda_x = E\left[Y_x\right]$$

where Y_x represents the number of photons that make it to depth x.

a) Write a differential equation which describes the behavior of λ_x as a function of x. (Hint: your differential equation should have the form $\frac{d\lambda_x}{dx}$ = something.)

b) Solve the differential equation to form an expression for λ_x in terms of $u(x)$ and λ_0.

c) Calculate an expression for the integral of the density, $\int_0^T u(x)dx$, in terms of λ_0 and λ_T.

d) In practice, how do you measure λ_0 and λ_T in a real CT system?

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Problem 2.(20pt)

Consider a deterministic gray level image, $g(m, n)$, which takes values on the interval $[0, 1]$, and let $T(m, n)$ be a random "white noise" threshold array of i.i.d. uniformly distributed random variables on the interval $[0, 1]$. Then the associated halftoned output image is given by

$$b(m,n) = \begin{cases} 1 & \text{if } g(m,n) \geq T(m,n) \\ 0 & \text{otherwise} \end{cases} .$$

Also, define the display error as $d(m, n) = b(m, n) - g(m, n)$.

a) Calculate simple expressions for the three quantities $P\{b(m, n) = 0\}$, $P\{b(m, n) = 1\}$, and $E[b(m, n)]$.

b) Calculate a simple expressions for the variance of $b(m, n)$.

c) Calculate the mean and variance of $d(m, n)$.

For parts d) through f), assume that $g(m, n) = g$, i.e., is a constant gray level g.

d) Calculate the autocovariance of $d(m, n)$ denoted by $R(k, l) = E[d(m, n)d(m + k, n + l)]$.

e) Calculate the power spectrum of $d(m, n)$ denoted by $S(e^{j\mu}, e^{j\nu})$.

f) Will $b(m, n)$ be a good quality halftone for the gray level g? Justify your answer.

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Problem 3.(20pt)

Consider a 2D Gaussian random vector, X, with distribution $N(0, R)$, and covariance $R = E\Lambda E^t$ where $\Lambda = diag\{16, 1\}$, and

$$E = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

a) Determine an orthonormal transformation T, so that $Y = TX$, where $Y = [Y_1, Y_2]^t$ are independent Gaussian random variables.

b) Determine a transformation H, so that $Z = HX$, where $Z = [Z_1, Z_2]^t$ are independent Gaussian random variables with mean 0 and variance 1.

c) For $\theta = \pi/3$, sketch a contour plot of the 2D distribution of X with X_1 as the horizontal axis and X_2 as the vertical axis and label critical dimensions of the sketch.

d) For $\theta = \pi/3$, sketch a contour plot of the 2D distribution of Y with Y_1 as the horizontal axis and Y_2 as the vertical axis and label critical dimensions of the sketch.

e) For $\theta = \pi/3$, sketch a contour plot of the 2D distribution of Z with Z_1 as the horizontal axis and Z_2 as the vertical axis and label critical dimensions of the sketch.

f) Why are these transformations useful?

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Problem 4.(20pt)

Consider an linear space invariant imaging system which can be modeled by

$$y = Ax + w \ ,$$

where x and y are the input and output images of the imaging system represented in raster order as a column vectors of length N, and w is a column vector of i.i.d. $N(0, \sigma^2)$ additive white noise. Furthermore, we will model x as a Gaussian random process with distribution $N(\mu_x, R_x)$.

a) Is any real input image a Gaussian random process? Justify your answer.

b) Are images accurately modeled by Gaussian random processes? Justify your answer.

For the following c) through g), assume that the image x is a Gaussian random process.

c) Calculate the mean and covariance of y denoted by μ_y and R_y.

d) Give an expression for the probability density of y.

e) Let $E[x|y] = f(y)$. Is $f(y)$ random or deterministic? What form must the function $f(y)$ have?

f) What does the answer to part e) above tell you about the form of the minimum mean squared error (MMSE) estimator of the image x given y?

g) In general, are linear filters good choices for removing noise from images? Justify your answer.

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Problem 5.(20pt)

Consider a Gaussian random variable $X \sim N(0, \sigma)$ with mean zero and variance σ^2. Then the rate $R(\delta)$ and the distortion $D(\delta)$ are functions of the parameter δ that represents the squared quantization step size. The functions are given by

$$R(\delta) = \max\left\{\frac{1}{2}\log\left(\frac{\sigma^2}{\delta}\right), 0\right\} \tag{1}$$

$$D(\delta) = \min\left\{\sigma^2, \delta\right\} \tag{2}$$

Furthermore, let $Z = [X_1, \cdots, X_N]$ be a row vector of independent Gaussian random variables with $X_n \sim N(0, \sigma_n^2)$ where $\sigma_{n-1}^2 \geq \sigma_n^2$.

a) Sketch the distortion-rate function for X. (Put distortion on the vertical axis and the rate on the horizontal axis.)

b) Explain the meaning of the distortion-rate function.

c) Calculate the mean and covariance of Z.

d) Write an expression for the rate and distortion of Z.

e) How can one calculate the distortion-rate function for a Gaussian vector Y with mean zero and covariance $R = E\Lambda E^t$ where $EE^t = I$ and Λ is diagonal?

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