Multivariate Gaussian Distribution

• Let x be a zero-mean random variable on \mathbb{R}^p

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}} |R|^{-1/2} \exp\{\mathbf{x}^T R^{-1} \mathbf{x}\}$$

where R is the $p \times p$ covariance matrix.

 \bullet The matrix R is a positive definite symmetric matrix, then

$$R = E\Lambda E^t$$

where $E = [\mathbf{e}_1, \dots, \mathbf{e}_p]$ is an orthonormal matrix of eigenvectors, and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_p)$ is a diagonal matrix of eigenvalues.

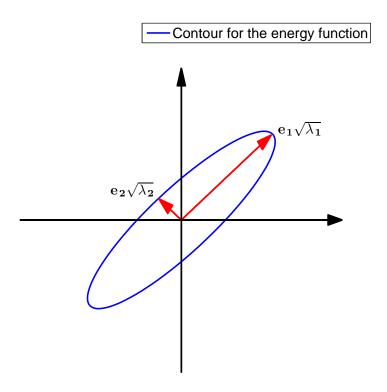
• Therefore, we have $E^tE = I$.

Contour for the Energy Function

• Intuitively, the energy function (square of the Mahalanobis distance)

$$f(\mathbf{x}) = \mathbf{x}^t R^{-1} \mathbf{x}$$

has contour plots shown here.



Gaussian Random Variable Decorrelation

• Consider $\tilde{\mathbf{x}} = E^t \mathbf{x}$, then

$$\mathbb{E}[\tilde{\mathbf{x}} \ \tilde{\mathbf{x}}^t] = \mathbb{E}[E^t \mathbf{x} \ \mathbf{x}^t E]$$

$$= E^t \mathbb{E}[\mathbf{x} \ \mathbf{x}^t] E$$

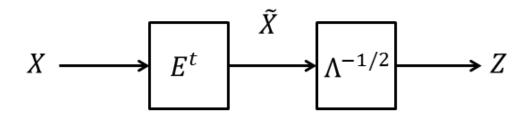
$$= E^t R E$$

$$= E^t E \Lambda E^t E$$

$$= \Lambda$$

- Therefore, the elements of $\tilde{\mathbf{x}}$ are uncorrelated with variance $\mathbb{E}[\tilde{x}_i^2] = \Lambda_{ii}$.
- The elements of $\tilde{\mathbf{x}}$ are independent, since $\tilde{\mathbf{x}}$, as the linear transform of \mathbf{x} , is Gaussian distributed.

Whitening Gaussian Random Variables



$$\mathbb{E}[Z|Z] = I$$

• So E^t decorrelates x, while $\Lambda^{-\frac{1}{2}}E^t$ whitens x.

$$E^t \mathbf{x} = \begin{bmatrix} \mathbf{e}_1^t \\ \vdots \\ \mathbf{e}_p^t \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix}$$

- The eigenvectors e_k , called eigen-signals, are basis vectors to represent the signal x.
- If x represents an image, then the eigenvectors e_k are also called *eigenimages*.

Eigenimage Estimation

• Assume we have n training vectors $X = [\mathbf{x}_1, \dots, \mathbf{x}_n]$, then

$$R_x = E[\mathbf{x}_k \ \mathbf{x}_k^t]$$

$$= E[XX^t]/n$$

$$\cong XX^t/n$$

$$= S$$

where $S = \frac{1}{n}XX^t$ is the sample correlation matrix

$$S_{ij} = \frac{1}{n} \sum_{k=1}^{n} X_{ik} X_{jk}$$

ullet Decompose S as

$$S = \hat{E}\hat{\Lambda}\hat{E}^t$$

where \hat{E} is an estimate of the eigenvectors, and $\hat{\Lambda}$ is an estimate of the eigenvalues.

ullet E could be very large, especially when X represents n images.

Singular Value Decomposition (SVD)

• For n < p it looks like

$$\begin{bmatrix} X \\ p \times n \end{bmatrix} = \begin{bmatrix} U \\ p \times n \end{bmatrix} \begin{bmatrix} \Sigma \\ n \times n \end{bmatrix} \begin{bmatrix} V^t \\ n \times n \end{bmatrix}$$

- The columns of ${\cal U}$ are orthonormal and called left hand singular vectors.
- The columns of V are orthonormal and called right hand singular vectors.
- Σ is diagonal matrix of singular values.

Eigenimage Estimation using SVD

Notice that

$$XX^{t} = U\Sigma V^{t}V\Sigma U^{t}$$
$$= U\Sigma^{2}U^{t}$$

So U is the set of the desired eigenvectors of XX^t .

- \bullet How to compute U?
 - Notice that $X^tX = V\Sigma^2V^t$ is a $n \times n$ matrix, and V contains eigenvectors of a much smaller matrix.
 - Algorithm
 - * Find eigenvectors V of the matrix X^tX
 - * Compute $XV = U\Sigma$, Then the columns of U are the normalized columns of XV, and Σ are the normalization factors.