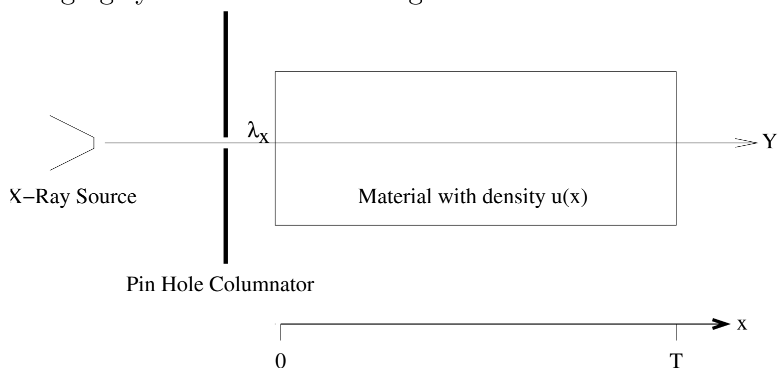
Topics: Tomography and MRI

Spring 2008 Exam 1: Problem 4 (tomography)

Consider an X-ray imaging system shown in the figure below.



Photons are emitted from an X-ray source and columnated by a pin hole in a lead shield. The columated X-rays then pass in a straight line through an object of length T with density $u(x)$ where x is the depth into the object. The number of photons in the beam at depth x is denoted by the Poisson random variable Y_x with $E[Y_x] = \lambda_x$ where all distances are measured in units of cm and all absorption constants are measured in units of cm^{-1}.

a) Write a differential equation which describes the behavior of λ_x as a function of x.

Solution:

The deduction in the number of photons can be expressed as: $d\lambda_x = -\lambda_x u(x)dx$.

$$\therefore \frac{d\lambda_x}{dx} = -\lambda_x u(x)$$

b) Solve the differential equation of part a)

Solution:

The solution to the equation given in a) is given by:

$$\lambda_x = \lambda_0 e^{-\int_0^x u(t)dt} \text{ , where } \lambda_0 \text{ is the initial dosage}$$

c) Specify how the photon counts Y_x can be used to compute the path integral of $u(x)$.

Solution:

From b), $e^{-\int_0^x u(t)dt} = \frac{\lambda_x}{\lambda_0}$, $\quad \therefore \int_0^x u(t)dt = -log\frac{\lambda_x}{\lambda_0} \approx -log\frac{Y_x}{\lambda_0}$, i.e., once the photon count Y_x is measured, the path integral of $u(x)$ is approximated to $-log\frac{Y_x}{\lambda_0}$.

Spring 2008 Final: Problem 5 (MRI)

Consider an MRI that only images in one dimension, x. So for example, the object being imaged might be a thin rod oriented along the x-dimension.

In this example, assume that the magnetic field strength at each location is given by

$$M_o + G(t)x$$

where M_o is the static magnetic field strength and $G(t)x$ is the linear gradient field in the x dimension. Then the frequency of precession for a hydrogen atom (in rad/sec) is given by the product of γ, the gyromagnetic constant, and the magnetic field strength.

a) Calculate $\omega(x,t)$, the frequency of precession of a hydrogen atom at location x and time t.

Solution:

$$\omega(x,t) = \gamma\left(M_0 + G(t)x\right)$$

b) Calculate $\phi(x,t)$, the phase of precession of a hydrogen atom at location x and time t assuming that $\phi(x,0) = 0$.

Solution:

$$\begin{aligned}
\frac{d\phi(x,t)}{dt} &= \omega(x,t) = \gamma\left(M_0 + G(t)x\right) \\
\Rightarrow \phi(x,t) &= \int_0^t \gamma\left(M_0 + G(\tau)x\right) d\tau \\
&= \int_0^t \gamma M_0 d\tau + \int_0^t \gamma G(\tau) x d\tau \\
&= \gamma M_0 t + x k_x(t) \text{ , where } k_x(t) = \textstyle\int_0^t \gamma G(\tau) d\tau
\end{aligned}$$

c) Calculate $r(x,t)$, the signal radiated from hydrogen atoms in the interval $[x, x+dx]$ at time t.

Solution:

$$r(x,t) = a(x)e^{j\phi(t)} = a(x)e^{j\gamma M_0 t}e^{jk_x(t)x}dx$$

d) Calculate $r(t)$, the signal radiated from hydrogen atoms along the entire object.

Solution:

$$r(t) = e^{j\gamma M_0 t}\int_{-\infty}^{\infty} a(x)e^{jk_x(t)x}dx = e^{j\gamma M_0 t}A\left(-k_x(t)\right) \text{ , where } A(\omega) = \mathcal{F}\left\{a(x)\right\}$$

e) Calculate an expression for $a(x)$, the quantity of processing hydrogen atoms along the thin rod, from the function $r(t)$.

Solution:

From (d), $r(t) = e^{j\gamma M_0 t} A\left(-k_x(t)\right)$, where $A(\omega) = \mathcal{F}\left(a(x)\right)$.

$$\begin{aligned} \Rightarrow a(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{j\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(-\omega) e^{-j\omega x} d\omega \end{aligned}$$

i.e, a(x) is obtained by the following steps:

1) Adjusting the values of $G(t)$ so that $\omega(t) \triangleq k_x(t) = \int_0^t \gamma G(\tau) d\tau$ gives from $-\omega_{max}$ to ω_{max}.
2) Record the values of $r(t)$ for different values of $\omega(t)$.
3) Compute $A(\omega(t)) = r(t) e^{-j\gamma M_0 t}$.
4) Using the calculated values for $A\left(-\omega(t)\right)$ to compute $a(x) = \dfrac{1}{2\pi} \int_{\omega_{max}}^{\omega_{max}} A(-\omega) e^{-j\omega x} d\omega$.

Spring 2008 Exam 1: Problem 2 (Tomography)

Assume that we know (or can measure) the function

$$p(x) = \int_{-\infty}^{\infty} f(x,y)dy \ .$$

Using the definitions of the Fourier transform, derive an expression for $F(u,0)$ in terms of the function $p(x)$.

Solution:

$$f(x,y) \overset{CSFT}{\Longleftrightarrow} F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$\begin{aligned}
\therefore F(u,0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dxdy \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)dy\right)e^{-j2\pi ux}dx \\
&= \int_{-\infty}^{\infty} p(x)e^{-j2\pi ux}dx, \text{ using the definition of } p(x) \\
&= P(u)
\end{aligned}$$

Therefore, $F(u,0)$ is equal to the CTFT of the function $p(x)$.