Topics: Entropy and lossless image coding

Spring 2008 Final: Problem 2 (entropy coding)

Let X_n be a discrete-time random process with i.i.d. samples, and distribution given by $P\{X_n = k\} = p_k$ where

$$(p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7) = \left(\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right)$$

a) What is the value of p_8? Why?

Solution:

Since $\sum_{k=0}^{7} p_k = 1$, if the discrete-time random process X_n takes the value $X_n = 8$, the associated probability $P(\{X_n = 8\}) = p_8$ should be zero. Therefore, $p_8 = 0$.

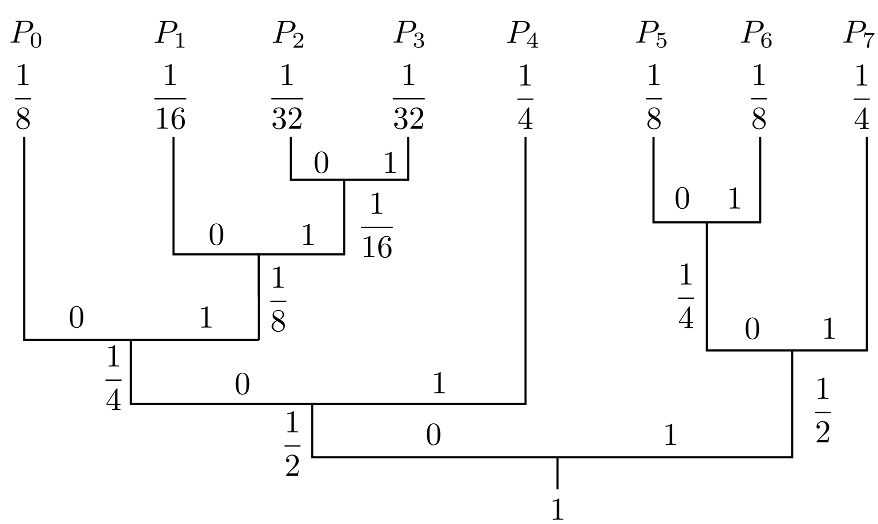
b) Calculate the entropy $H(X_n)$ in bits.

Solution:

$$\begin{aligned}
H(X_n) &= -\sum_{k=0}^{7} p_k \log_2 p_k \\
&= \frac{1}{8} \times 3 + \frac{1}{16} \times 4 + \frac{1}{32} \times 5 + \frac{1}{32} \times 5 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 3 + \frac{1}{4} \times 2 \\
&= \frac{43}{16} \text{ bits}
\end{aligned}$$

c) Draw the Huffman tree and determine the binary Huffman code for each possible symbol.

Solution:



p_k	Symbol $X_n = k$	Huffman Code	length l_k (in bits)
1/8	0	000	3
1/16	1	0010	4
1/32	2	00110	5
1/32	3	00111	5
1/4	4	01	2
1/8	5	100	3
1/8	6	101	3
1/4	7	11	2

d) Calculate the expected code length per symbol.

Solution:

$$\begin{aligned}\text{expected length} &= \sum_{k=0}^{7} p_k l_k \\ &= \frac{1}{8} \times 3 + \frac{1}{16} \times 4 + \frac{1}{32} \times 5 + \frac{1}{32} \times 5 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 3 + \frac{1}{4} \times 2 \\ &= \frac{43}{16} \text{ bits} = H(X_n)\end{aligned}$$

e) Are there better codes for X_n?

If so, what are they? If not, why not?

Solution:

Since the entropy for this source is $H(X_n) = \frac{43}{16}$ bits, which is the lower bound of bitrate to represent the source data, for any coder, we'll expect to have length $L \geq H$.

The Huffman Coder achieves the lower bound since $L = H$.

$\Rightarrow$ No other coder will do better. Huffman Coder is optimal for this source data.

Spring 2005 Final: Problem 1 (lossless image coding)

Consider a lossless predictive coder which predicts the pixel $X_{s_1,s_2} = k$ from the two pixels $X_{s_1,s_2-1} = i$ and $X_{s_1-1,s_2} = j$. In order to design the predictor, you first measure the histogram for the values of i, j, k from some sample images. This results in the following measurements.

i	j	k	$h(i,j,k)$
0	0	0	30
0	0	1	2
0	1	0	4
0	1	1	12
1	0	0	12
1	0	1	4
1	1	0	2
1	1	1	30

a) Use the values of $h(i,j,k)$ to calculate $\hat{p}(k|i,j)$, an estimate of

$$p(k|i,j) = P\{X_{s_1,s_2} = k | X_{s_1,s_2-1} = i,\, X_{s_1-1,s_2} = j\}$$

and use them to fill in the table below.

Solution:

i	j	k	$\hat{p}(k|i,j)$
0	0	0	30/32
0	0	1	2/32
0	1	0	8/32
0	1	1	24/32
1	0	0	24/32
1	0	1	8/32
1	1	0	2/32
1	1	1	30/32

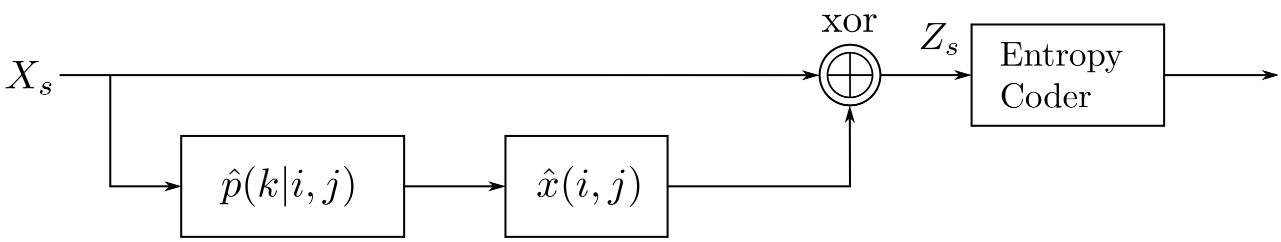
b) For each value of i, and j, compute a binary valued estimate of X_{s_1,s_2} and use it to fill in the table below.

Solution:

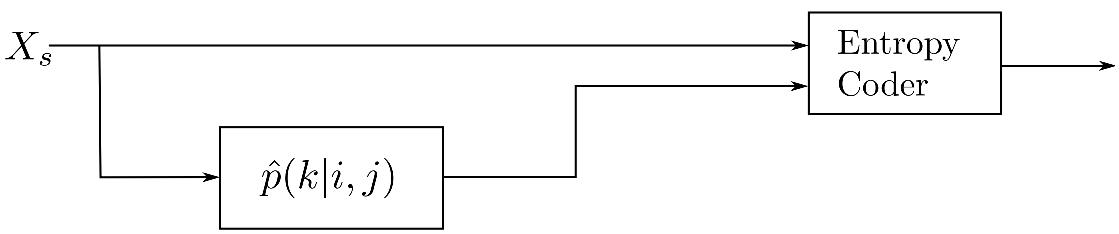
i	j	$\hat{X}_{s_1,s_2}$
0	0	0
0	1	1
1	0	0
1	1	1

c) Draw a block diagram for the lossless predictive coder. The block diagram should include an entropy coder.

Solution 1:



Solution 2:



d) Assuming the prediction errors are independent (but not identically distributed), calculate an expression for the theoretically achievable bit rate for the lossless predictive encoder. Justify your answer. (Hint: Use the fact that $\log_2(3) = 1.585$, $\log_2(7) = 2.807$, and $\log_2(15) = 3.907$.)

Solution 1:

$Z_s = X_s \oplus \hat{X}_s$, where $\oplus$ is the xor operator

$$\begin{aligned}
P\{Z_s = 1\} &= P\{i=0, j=0, k=1\} + P\{i=0, j=1, k=0\} \\
&\quad + P\{i=1, j=0, k=1\} + P\{i=1, j=1, k=0\} \\
&= \frac{2}{96} + \frac{4}{96} + \frac{4}{96} + \frac{2}{96} \\
&= \frac{12}{96} = \frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
H(Z_s) &= -\frac{1}{8}\log_2\frac{1}{8} - \frac{7}{8}\log_2\frac{7}{8} \\
&= \frac{3}{8} + \frac{7}{8}3 - \frac{7}{8}\log_2 7 \\
&= \frac{3}{8} + \frac{7}{8}(3 - 2.81) = \frac{3}{8} + 0.1685 \\
&= 0.54 \text{ bits per pixel}
\end{aligned}$$

Solution 2:

$$
\begin{aligned}
&\frac{1}{3}H\left(\frac{2}{32}\right)+\frac{1}{6}H\left(\frac{8}{32}\right)+\frac{1}{6}H\left(\frac{8}{32}\right)+\frac{1}{3}H\left(\frac{2}{32}\right)\\
&=\frac{1}{3}\left(H\left(\frac{2}{32}\right)+H\left(\frac{8}{32}\right)+H\left(\frac{2}{32}\right)\right)\\
&=\frac{2}{3}H\left(\frac{2}{32}\right)+\frac{1}{3}H\left(\frac{8}{32}\right)\\
&=\frac{2}{3}H\left(\frac{1}{16}\right)+\frac{1}{3}H\left(\frac{1}{4}\right)\\
&=0.49529
\end{aligned}
$$

$$
\begin{aligned}
H\left(\frac{1}{16}\right)&=\frac{1}{16}\log_2 16+\frac{15}{16}\log_2\frac{16}{15}\\
&=\frac{1}{4}+\frac{15}{16}(4-\log_2(15))\\
&=\frac{1}{4}+\frac{15}{16}(5-3.907)\\
&=\frac{1}{4}+\frac{15}{16}(0.093)\\
&=0.3373
\end{aligned}
$$

$$
\begin{aligned}
H\left(\frac{1}{4}\right)&=\frac{1}{4}\log_2 4+\frac{3}{4}\log_2\frac{4}{3}\\
&=\frac{1}{2}+\frac{3}{4}(2-\log_2 3)\\
&=\frac{1}{2}+\frac{3}{4}(0.4150)\\
&=0.8113
\end{aligned}
$$

Spring 2001 Final: Problem 1 (entropy coding)

Let X_n be a discrete random variable which takes values on the set $\{0, 1, \cdots, 5\}$, and let

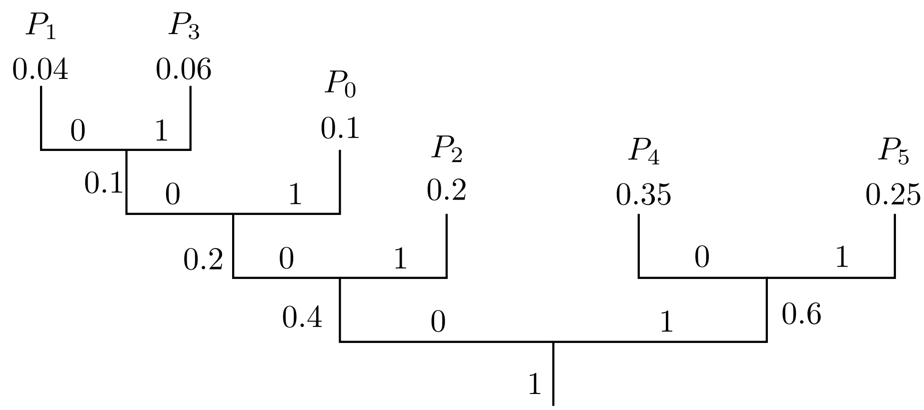
$$P\{X_n = k\} = p_k$$

where

$$(p_0, p_1, p_2, p_3, p_4, p_5) = (0.1, 0.04, 0.2, 0.06, 0.35, 0.25)$$

a) Draw and fully label the binary tree used to form a Huffman code for X_n.

Solution:



b) Write out the Huffman codes for the six symbols $0, 1, \cdots, 5$

Solution:

Symbol $X_n = k$	Huffman Code	p_k	length l_k (in bits)
0	001	0.1	3
1	0000	0.04	4
2	01	0.2	2
3	0001	0.06	4
4	10	0.35	2
5	11	0.25	2

c) Compute the expected code length for your Huffman code.

Solution:

$$\begin{aligned}\text{expected length} &= \sum_{k=0}^{5} p_k l_k = 0.1 \times 3 + 0.04 \times 4 + 0.2 \times 2 + 0.06 \times 4 + 0.35 \times 2 + 0.25 \times 2 \\ &= 2.3 \text{ bits per symbol}\end{aligned}$$