Topics: Nonlinear filtering

Spring 2007 Final: Problem 5 (nonlinear filtering)

Consider an signal $Y_n = S_n + W_n$ where S_n is a unknown signal, W_n is i.i.d. Gaussian noise with mean 0 and variance 1.

Your job (should you choose to accept it) is to recover a good estimate of S_n by applying a function to a 5 point window about the location n. So the estimate is given by

$$\hat{S}_n = f(Z_n)$$

where

$$Z_n = [Y_{n-2}, Y_{n-1}, Y_n, Y_{n+1}, Y_{n+2}]^t \ .$$

a) Assuming that $S_n = \mu$, where μ is a constant, then what is a good choice for the function $f(\cdot)$? Justify your answer.

Solution:

$\widehat{S}_n = \frac{1}{5}\sum_{k=-2}^{2} Y_{n+k}$ would be an appropriate choice. It should be noted that $E\left[\widehat{S}_n\right] = \mu$, and $\widehat{S}_n$ is the MMSE estimator of μ based on points in the local window.

b) Assuming that S_n is a slowly varying function of n, then what is a good choice for the function $f(\cdot)$? Justify your answer.

Solution:

A low pass smoothing kernel that gives more emphasis to points that are close to the center for the local window would be an appropriate choice.

$$\therefore \widehat{S}_n = \sum_{k=-2}^{2} g_k Y_{n+k}, \text{ where } g_k \text{ could be selected as } g_k = \begin{cases} ce^{-\frac{k^2}{\sigma^2}} & |k| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

c) Assuming the S_n has the form

$$S_n = a_n + (10)b_n$$

where a_n is a slowly varying function of n, and b_n is i.i.d. with discrete probability density function $P\{b_n = 1\} = P\{b_n = -1\} = 0.001$ and $P\{b_n = 0\} = 0.998$, then what is a good choice for the function $f(\cdot)$? Justify your answer.

Solution:

$\widehat{S}_n$ could be estimated using the following algorithm:

$$\begin{aligned}
\text{Define } K &= \{-2, -1, 0, 1, 2\} \\
k_m &= \arg\min_{k \in K}(Y_{n+k}) \\
k_M &= \arg\max_{k \in K}(Y_{n+k}) \\
R_n &= \frac{1}{3} \sum_{k \in K \setminus \{k_m, k_M\}} Y_(n+k)
\end{aligned}$$

$$\begin{aligned}
&\text{if } [(k_m == 0 \;||\; k_M == 0) \;\&\&\; |R_n - Y_n| > T] \\
&\qquad \widehat{S}_n = Y_n \\
&\text{else} \\
&\qquad \widehat{S}_n = R_n
\end{aligned}$$

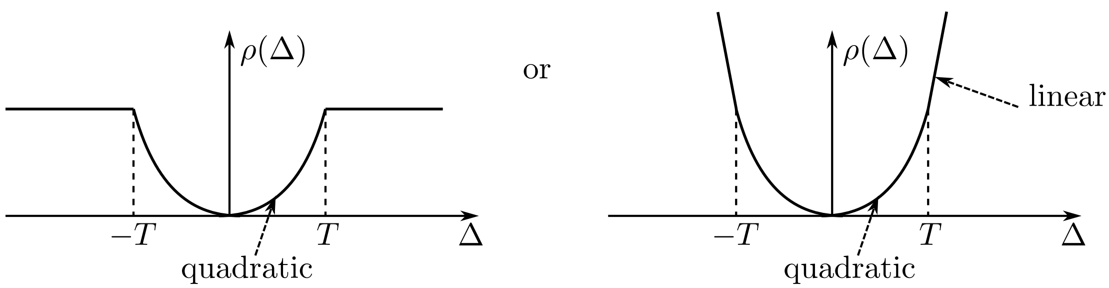
d) Assuming the S_n has the form

$$S_n = a_n + 10 \sum_{k=-\infty}^{\infty} b_k \text{pulse}_{10}(n-k)$$

where a_n is a very slowly varying function of n, and b_n is i. i. d. with discrete probability density function $P\{b_n = 1\} = P\{b_n = -1\} = 0.001$ and $P\{b_n = 0\} = 0.998$, then what is a good choice for the function $f(\cdot)$? [1] Justify your answer.

Solution:

$$\widehat{S}_n = \arg\min_{\theta} \sum_{k=-2}^{2} \rho(|\theta - Y_{n+k}|),\ \rho(\Delta) \text{ could be selected as follows:}$$



[1] Note that $\text{pulse}_N(n) = u(n) - u(n-N)$.

Spring 2006 Final: Problem 4 (M-estimators)

Consider the set of data $\{x_n\}_{n=0}^{N-1}$ for N odd. We would like to estimate a "central value" using a method known as M-estimation. To do this we compute the following function

$$\hat{\theta} \quad = \quad \arg\min_{\theta} \left\{ \sum_{n=0}^{N-1} \rho(x_n - \theta) \right\}$$

where ρ is a function with the properties that $\rho(\Delta) \geq 0$ and $\rho(-\Delta) = \rho(\Delta)$.

a) What function $\rho(\Delta)$ will result in the mean as shown below?

$$\text{mean is } \hat{\theta} = \frac{1}{N} \sum_{i=0}^{N-1} x_i$$

Solution:

$$\rho(\Delta) = |\Delta|^2$$

b) What function $\rho(\Delta)$ will result in the median?

Solution:

$$\rho(\Delta) = |\Delta|$$

c) Select a function $\rho(\Delta)$ which usually produces an estimate close to the mean, but limits the influence of a single value of x_i.

Solution:

$$\rho(\Delta) = \begin{cases} |\Delta|^2 & \text{for } |\Delta| < T \\ |2T\Delta| - T^2 & \text{for } |\Delta| > T \end{cases}$$