EE 637 Final

May 3, Spring 2017

Name: ______________________________

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \stackrel{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \stackrel{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \stackrel{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \stackrel{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t) y(t) \stackrel{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \stackrel{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \stackrel{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \stackrel{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \stackrel{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\text{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\text{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\text{comb}_T\left[x(t)\right] \stackrel{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\text{rep}_T\left[x(t)\right] \stackrel{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(f) &= Y\left(e^{j2\pi f T}\right) \end{aligned}$$

Name: ______________________________

Problem 1.(20pt)

Let $s(x,y) = \text{sinc}(\alpha x, \beta y)$ where x and y have units of inches and α and β are positive constants with units of inch^{-1}. Furthermore, let $g(m,n) = s(mA, nB)$ be a discrete-space function where A and B are the sampling intervals in the x and y directions, respectively, in inches.

a) Calculate $S(u,v)$, the CSFT of $s(x,y)$.

b) Give the cuttoff frequencies of $s(x,y)$ in both x and y directions in units of cycles per inch.

c) Specify the Nyquist sampling rates in units of cycles per inch along the x and y directions that are required to ensure perfect reconstruction. Also, specify constraints on the associated values of A and B in terms of α and β.

d) Calculate $G(e^{j\mu}, e^{j\nu})$, the DSFT of $g(m,n)$.

e) Sketch the function $G(e^{j\mu}, e^{j\nu})$ on the region $(\mu,\nu) \in (-\pi,\pi) \times (-\pi,\pi)$ when $A = B = 1$ and $\alpha = \beta = \frac{1}{2}$.

(a) $S(u,v) = \dfrac{1}{\alpha\beta} \text{rect}\left(\dfrac{u}{\alpha}, \dfrac{v}{\beta}\right)$

(b) $u_c = \dfrac{\alpha}{2}$ $\quad v_c = \dfrac{\beta}{2}$

(c) $u_N = \alpha$ $\quad v_N = \beta$

sampling rate must be greater than or equal to Nyquist frequencies

$\dfrac{1}{A} > \alpha$, $\quad \dfrac{1}{B} > \beta$.

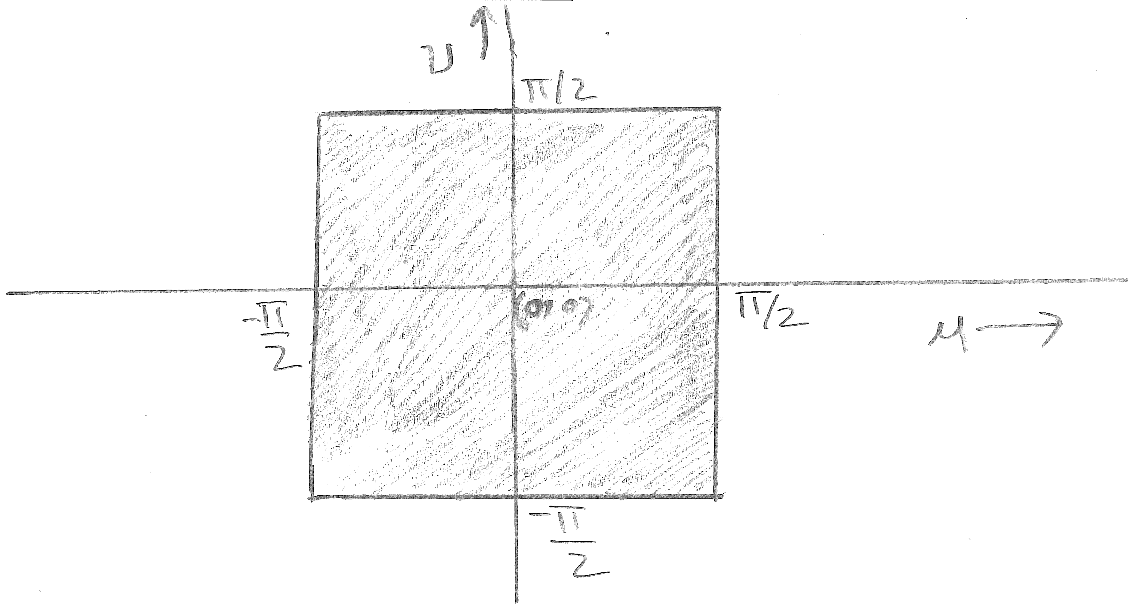
(d) $G(e^{j\mu}, e^{j\nu}) = \dfrac{1}{\alpha\beta} \dfrac{1}{AB} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \text{rect}\left(\dfrac{\mu - 2\pi k}{2\pi\alpha A}, \dfrac{\nu - 2\pi l}{2\pi\beta B}\right)$

(e) When $A = B = 1$ and $\alpha = \beta = \frac{1}{2}$, then

$G(e^{j\mu}, e^{j\nu}) = 4 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \text{rect}\left(\dfrac{\mu - 2\pi k}{\pi}, \dfrac{\nu - 2\pi l}{\pi}\right)$

$G(e^{j\mu}, e^{j\nu}) = 4$ in shaded region

Name: ______________________________



Name: ______________________

Name: ____________________

Problem 2.(20pt)

Consider a focal plane array with detector cell sizes $T \times T$, where T has units of inches. Let $g(x, y)$ be the incoming light signal, and let $s(m, n)$ be the measurement from the $(m, n)^{th}$ detector cell. The measurements $s(m, n)$ are related to the input light signal as

$$s(m,n) = \int_{\Re^2} h(x - Tm, y - Tn) g(x, y)\, dx\, dy \ ,$$

where $h(x, y) = rect\left(\frac{x}{T}, \frac{y}{T}\right)$. Furthermore, define

$$\tilde{g}(x, y) = h(x, y) * *g(x, y) \ ,$$

where $**$ represents 2-D convolution.

a) Show that $s(m, n) = \tilde{g}(Tm, Tn)$, and use this result to draw a block-diagram of a signal-processing model of the focal-plane array.

b) Calculate an expression for $\widetilde{G}(u, v)$ the CSFT of $\tilde{g}(x, y)$ in terms of $G(u, v)$.

c) Calculate an expression for $S(e^{j\mu}, e^{j\nu})$ the DSFT of $s(m, n)$ in terms $G(u, v)$.

d) What is the maximum spatial frequency that the sampling system can reconstruct based on Nyquist theory?

e) Calculate the frequency response of a linear space-invariant digital filter, $P(e^{j\mu}, e^{j\nu})$ that can compensate for the effect of the detector cell.

(a) $\tilde{g}(x,y) = h(x,y) ** g(x,y)$

$$\Rightarrow \tilde{g}(\xi,\eta) = \int_{\mathbb{R}^2} h(\xi - x, \eta - y)\, g(x,y)\, dx\, dy$$

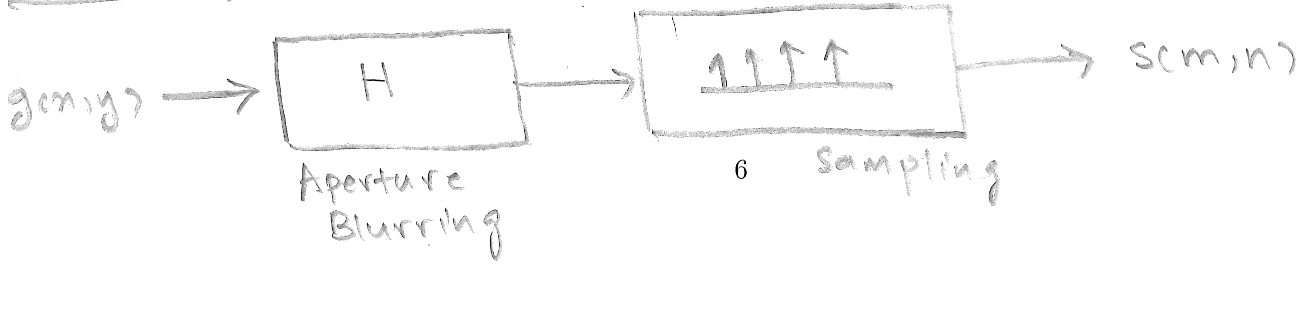
$$= \int_{\mathbb{R}^2} h(x - \xi, y - \eta)\, g(x,y)\, dx\, dy \quad \because h \text{ is symmetric.}$$

Now let's sample $\tilde{g}(\xi,\eta)$ at $\xi = mT$, $\eta = nT$.

$$\Rightarrow \tilde{g}(mT, nT) = \int_{\mathbb{R}^2} h(x - mT, y - nT)\, g(x,y)\, dx\, dy$$

$$= s(m,n).$$

Block Diagram



Name: ______________________

(b) $\tilde{G}(u,v) = H(u,v)\, G(u,v)$

$$= T^2 \operatorname{sinc}(uT, vT)\, G(u,v)$$

(c) $$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \tilde{G}\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right)$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\mu - 2\pi k}{2\pi}, \frac{\nu - 2\pi l}{2\pi}\right) G\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right)$$

(d) Maximum spatial frequency that can be reconstructed is $\frac{1}{T}$ cycles per inch in both horizontal and vertical directions.

(e) $$P(e^{j\mu}, e^{j\nu}) = \frac{1}{\operatorname{sinc}\left(\frac{\mu}{2\pi}, \frac{\nu}{2\pi}\right)}$$

This filter would compensate for the roll-off.

Name: ______________________________

Name: ______________________________

Problem 3.(20pt)

Let $Y \in \Re^N$ be a vector containing the pixels in an image window. We can model Y as

$$Y = \mu X + W$$

where $\mu \in \Re^N$ is a deterministic column vector of length N, $X \in \Re$ is scalar valued Gaussian random variable with mean 0 and variance σ^2, and $W \in \Re^N$ is a Gaussian random vector of i.i.d. noise components, each with distribution $N(0, \sigma_w)$.

Intuitively, Y is composed of a signal μ modulated by an amplitude X obscured by additive noise W. Our objective is to estimate the value of X from the observations Y. To do this, we will form a MMSE linear estimator for S given by

$$\hat{X} = Y^t \theta$$

where $\theta \in \Re^N$ is a vector of coefficients.

Furthermore, define the covariance matrix of W given by

$$R_w = E\left[WW^t\right] ,$$

the covariance matrix of Y given by

$$R_y = E\left[YY^t\right] ,$$

and the cross-covariance column vector of Y and X given by

$$b = E\left[YX\right] .$$

a) Give a simplified expression for R_w in terms of σ_w.

b) Give a simplified expression for R_y in terms of σ_w, σ and μ.

c) Give a simplified expression for b in terms of σ_w, σ and μ.

d) Calculate an expression for the MSE given by $E\left[||X - \hat{X}||^2\right]$ in terms of σ^2, σ_w^2, θ, and μ.

e) Use the above results to calculate a closed form expression for $\hat{X}$.

(a) $R_w = E[WW^t] = \sigma_w^2 I_N$ where I_N is $N \times N$ identity matrix.

(b) $R_y = E[YY^t]$

$= E[(\mu X + W)(\mu X + W)^t]$

$= E[\mu\mu^t X^2 + 2\mu W^t X + WW^t]$

$= \mu\mu^t E[X^2] + 2\mu E[W^t X] + E[WW^t]$

Name: ____________________

Since W and X are independent, so

$E[W^t X] = E[W^t]\, E[X] = 0$

$R_y = \mu\mu^t \sigma^2 + \sigma_w^2 I$

(c) $b = E[YX]$

$= E[\mu X^2 + XW]$

$= \mu E[X^2] + E[X] E[W]$

$= \mu \sigma^2$

(d) $E[\|X - \hat{X}\|^2] = E[(X - Y^t\theta)^2]$

$= E[X^2 - 2\theta^t Y X + \theta^t Y Y^t \theta]$

$= E[X^2] - 2\theta^t E[YX] + \theta^t E[YY^t]\theta$

$= \sigma^2 - 2\theta^t b + \theta^t R_y \theta$

$= \sigma^2 - 2\theta^t \mu \sigma^2 + \theta^t(\mu\mu^t\sigma^2 + \sigma_w^2 I)\theta$

(e) $\frac{\partial}{\partial\theta} E[\|X - \hat{X}\|^2] = -2b + 2R_y\theta = 0$.

$\Rightarrow \theta = R_y^{-1} b$

$\Rightarrow \theta = (\mu\mu^t\sigma^2 + \sigma_w^2 I)^{-1} \mu \sigma^2$

Note: R_y is positive definite because it's the sum of a

Name: ______________________

positive definite matrix $\sigma_w^2 I$ and a positive semidefinite matrix $\mu\mu^t \sigma^2$. Therefore Θ is the unique global minimizer of $E[\|X-\hat{X}\|^2]$.

Since $\hat{X} = Y^T \Theta$, so

$$\hat{X} = Y^T R_y^{-1} b.$$

Name: ______________________________

Problem 4.(20pt)

Consider an image $f(x,y)$ with a CSFT given by $F(u,v)$ and a forward projection

$$\begin{aligned} p_\theta(r) &= \mathcal{FP}\{f(x,y)\} \\ &= \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta))\, dz \ . \end{aligned}$$

a) Calculate the projections $p_\theta(r)$ of the function $f(x,y) = \delta(x,y)$.

b) Calculate the projections $p_\theta(r)$ of the function $f(x,y) = \delta(x-1,y)$.

c) Calculate the projections $p_\theta(r)$ of the function $f(x,y) = \mathrm{circ}(x,y)$.

d) Calculate the projections $p_\theta(r)$ of the function $f(x,y) = \mathrm{circ}(x-1,y)$.

(a) Since $\delta(x,y)$ is rotationally invariant, we shall find $p_0(r)$

$$p_\theta(r) = p_0(r) = \int_{-\infty}^{\infty} \delta(r,z)\, dz = \int_{-\infty}^{\infty} \delta(r)\,\delta(z)\, dz$$

$$= \delta(r).$$

(b) Let's put $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-1 \\ y \end{bmatrix}$.

$$p_\theta(r') = \int_{-\infty}^{\infty} f\left(A\begin{bmatrix} r' \\ z' \end{bmatrix}\right) dz$$

However $\begin{bmatrix} r' \\ z' \end{bmatrix} = A^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} + A^{-1}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} r \\ z \end{bmatrix} - \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

$\Rightarrow p_\theta(r') = p_\theta(r - \cos\theta)$.

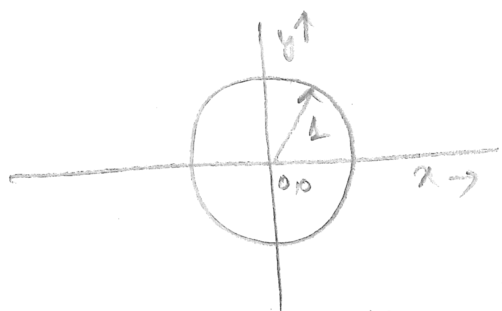
So when $f(x,y) = \delta(x-1,y) \Rightarrow p_\theta(r) = \delta(r - \cos\theta)$.

Name: ______________________________

So if $f(x,y) = \delta(x-1,y)$

$$\Rightarrow p_\theta(r) = \delta(r - \cos\theta).$$

(c) $$\text{circ}(x,y) = \begin{cases} 1 & \text{when } x^2+y^2 \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$



This is also rotationally invariant, so

$$p_\theta(r) = p_0(r) = \int_{-\sqrt{\frac{1}{4}-r^2}}^{\sqrt{\frac{1}{4}-r^2}} 1 \cdot dz \qquad \text{when } |r| \le 1/2$$

$$= 2\sqrt{\frac{1}{4}-r^2} \qquad \text{when } |r| \le 1/2$$

$$p_\theta(r) = \begin{cases} 2\sqrt{\frac{1}{4}-r^2} & \text{when } r \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(d) Using result of part (b)

$$p_\theta(r) = \begin{cases} 2\sqrt{\frac{1}{4} - (r-\cos\theta)^2} & |r-\cos\theta| \le 1/2 \\ 0 & |r-\cos\theta| > 1/2 \end{cases}$$

Name: ______________________________

Problem 5.(20pt)

Let $x(t)$ be a continuous-time signal, and $y(n) = x(Tn)$ be the associated discrete-time sampled signal.

a) Give an expression for the discrete approximation to

$$\left.\frac{dx(t)}{dt}\right|_{t=Tn}$$

the derivative of $x(t)$ at time $t = Tn$ in terms of the sampled signal $y(n)$.

b) Give an expression for the discrete approximation to

$$\left.\frac{d^2x(t)}{dt^2}\right|_{t=Tn}$$

the second derivative of $x(t)$ at time $t = Tn$ in terms of the sampled signal $y(n)$.

c) Give a general condition for detecting an edge in the signal based on the first discrete derivative.

d) Give a general condition for detecting an edge in the signal based on a combination of the first and second discrete derivatives.

e) What is the advantage of using both the first and second derivatives?

(a) $$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t-T)}{T}$$

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} \approx \frac{x(nT) - x(nT-T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

(b) $$\frac{d^2x(t)}{dt} \approx \frac{x'(t+T) - x'(t)}{T}$$

$$\left.\frac{d^2x(t)}{dt}\right|_{t=nT} \approx \frac{\frac{x(nT+T) - x(nT)}{T} - \frac{x(nT) - x(nT-T)}{T}}{T}$$

Name: ______________________

$$= \frac{y(n+1) - y(n) - y(n) + y(n-1)}{T^2}$$

$$= \frac{-2y(n) + y(n-1) + y(n+1)}{T^2}$$

(c) Condition based on 1^{st} derivative for edge detection:

$|y(n) - y(n-1)| > K$ where K is threshold.

(d) conditions based on 1^{st} + 2^{nd} derivative for edge detection:

Let $f_1(n) = y(n) - y(n-1)$ and $f_2(n) = -2y(n) + y(n-1) + y(n+1)$

<u>Conditions</u>

(1) $|f_1(n)| > K$

(2) $f_2(n-1)\, f_2(n+1) < 0$

where K_1 and K_2 are thresholds.

(e) By using both derivatives, we would have less false alarms in flat regions.

Name: ______________________________