EE 637 Final

May 5, Spring 2015

Name: Key

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}} [X(f)]$$

$$\mathrm{rep}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T} \right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

Name: ______________________________

Problem 1.(20pt)

Consider a color imaging device that takes input values of (r, g, b) and produces output (X, Y, Z) values given by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = A \begin{bmatrix} r^\alpha \\ g^\alpha \\ b^\alpha \end{bmatrix} .$$

where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} .$$

a) Calculate the white point of the device in chromaticity coordinates.

b) Determine the chromaticity coordinates of the three primaries associated with the r, g, and b components.

c) What is the gamma of the device?

a) White point corresponds to $r=1$, $g=1$ and $b=1$.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+c \\ d+e+f \\ g+h+i \end{bmatrix}$$

Let $C = a+b+c+d+e+f+g+h+i$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{a+b+c}{C} \\ \frac{d+e+f}{C} \\ \frac{g+h+i}{C} \end{bmatrix}$$

b) primary associated with r is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{a}{a+d+g} \\ \frac{d}{a+a+g} \\ \frac{g}{a+d+g} \end{bmatrix}$

primary associated with g is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{b}{b+e+h} \\ \frac{e}{b+e+h} \\ \frac{h}{b+e+h} \end{bmatrix}$

primary associated with b is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{c}{c+f+i} \\ \frac{f}{c+f+i} \\ \frac{i}{c+f+i} \end{bmatrix}$

c) The gamma of the device is α.

Name: ______________________________

Name: ______________________________

Problem 2.(20pt)

Let X be a scalar random variable and Z be a $M \times 1$ random vector. Assume that X and Z are zero mean with

$$\begin{aligned} b &= E[XZ] \\ R &= E[ZZ^t] \,. \end{aligned}$$

Further define the estimator

$$\hat{X} = \theta Z$$

where θ is a $1 \times M$ parameter vector.

a) **Derive** an expression for the value of θ that results in the minimum mean squared error **linear** estimator of X given Z.

b) What estimator always results in the minimum mean squared error estimate of X given Z?

c) Is the minimum mean squared error estimator of X always a linear function of Z? Justify your answer.

d) What estimator, $\hat{X} = T(Z)$, minimizes the following expression?

$$E[|X - \hat{X}|]$$

a) $\theta^* = \arg\min_\theta E[|X-\hat{X}|^2]$

$= \arg\min_\theta E[(X-\theta Z)(X-\theta Z)^t]$

$= \arg\min_\theta [X^2 - 2\theta b - \theta R \theta^t]$

$\frac{\partial}{\partial \theta}[X^2 - 2\theta b - \theta R \theta^t] = 0$

$\theta^* = b^t R^{-1}$

b) $E[X|Z]$

c) No. $E[X|Z]$ is a linear function of Z, if X and Z are jointly Gaussian. However, this does not hold in general.

d) Median.

Name: ______________________________

Name: ______________________________

Problem 3.(20pt)

Consider the 1-D error diffusion algorithm specified by the equations

$$\begin{aligned} b_n &= Q(y_n) \\ e_n &= y_n - b_n \\ y_n &= x_n + e_{n-1} \end{aligned}$$

where x_n is the input, b_n is the output, and $Q(\cdot)$ is a binary quantizer with the form

$$Q(y) = \begin{cases} 1 & \text{if } y > 0.5 \\ 0 & \text{if } y \le 0.5 \end{cases} .$$

where we assume that $e_0 = 0$ and the algorithm is run for $n \ge 1$.

Furthermore, define $d_n = x_n - b_n$.

a) Draw a flow diagram for this algorithm. Make sure to label all the signals in the flow diagram using the notation defined above.

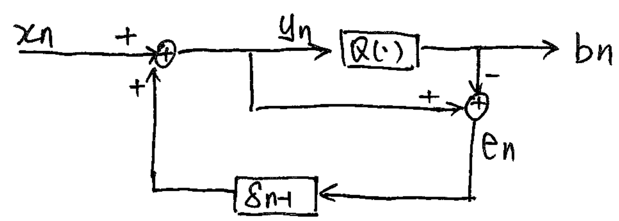
b) Calculate b_n for $n = 1$ to 10 when $x_n = 0.25$ and $e_0 = 0$.

c) Calculate an expression for d_n in terms of the quantization error e_n.

d) Calculate an expression for $\sum_{n=1}^{N} d_n$ in terms of the quantization error e_n.

e) What does the result of part d) above tell you about the output of error diffusion?

a)



b)

n	1	2	3	4	5	6	7	8	9	10
x_n	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
y_n	0.25	0.5	0.75	0	0.25	0.5	0.75	0	0.25	0.5
b_n	0	0	1	0	0	0	1	0	0	0
e_n	0.25	0.5	-0.25	0	0.25	0.5	-0.25	0	0.25	0.5

c) $d_n = x_n - b_n = x_n - (y_n - e_n) = x_n - (x_n + e_{n-1} - e_n) = e_n - e_{n-1}$

d) $\sum_{n=1}^{N} d_n = \sum_{n=1}^{N} (e_n - e_{n-1}) = e_N - e_0 = e_N$

Name: ______________________________

e) ① It tells us that the accumulative ~~error~~ is bounded.

Now we prove that e_N is bounded.

1) when $N=0$ $e_0=0$ is bounded.

2) We assume the e_k is bounded, then.

$y_{k+1} = x_{k+1} + e_k$ is bounded since x_{k+1} is bounded as well.

Then $e_{k+1} = y_{k+1} - b_{k+1}$ is bounded, since y_{k+1} is bounded

and $b_{k+1} = Q(y_{k+1})$ is bounded.

Thus, e_k is bounded $\Rightarrow$ e_{k+1} is bounded.

3) By induction. e_N is bounded for $N \in Z^+$.

② It tells us that the local average of the signal is approximately maintained.

$$\sum_{n=1}^{N} d_n = \sum_{n=1}^{N} (x_n - b_n) = e_N$$

Then $$\sum_{n=1}^{N} b_n = \sum_{n=1}^{N} x_n - e_N$$

$$\Rightarrow \frac{1}{N}\sum_{n=1}^{N} b_n = \frac{1}{N}\sum_{n=1}^{N} x_n - \frac{1}{N}e_N$$

So the local average of the input is approximately the same as the local average of the output.

Name: ________________________________

Problem 4.(20pt)

Consider a discrete-time random process X_n with i.i.d. samples that are Gaussian with mean 0 and variance $\sigma^2 > 0$.

The rate distortion relation for this source is then given by

$$R(\Delta) = \max\left\{\frac{1}{2}\log_2\left(\frac{\sigma^2}{\Delta^2}\right), 0\right\}$$
$$D(\Delta) = \min\left\{\sigma^2, \Delta^2\right\}$$

a) Plot the minimum possible rate (y-axis) versus distortion (x-axis) required to code this source when $\sigma^2 = 1$.

b) Explain the meaning of the distortion-rate function.

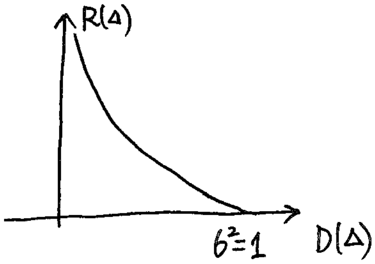
c) How many bits per sample are required in order to achieve zero distortion?

d) How many bits per sample are required in order to achieve a distortion of $D = \sigma^2$?

e) Let $\{X_n\}_{n=1}^N$ be a set of independent Gaussian random variables each with zero mean and variance σ_n^2 where $\sigma_{n-1}^2 \geq \sigma_n^2$. Write an expression for the rate and distortion.

f) Calculate the distortion-rate function for a Gaussian vector X with mean zero and covariance $R = E\Lambda E^t$ where $EE^t = I$ and Λ is diagonal?

a)



b) It is the lower bound on achievable rates at a given distortion. Given $D' > D(\Delta)$ and $R' > R(\Delta)$, one can find a coder that achieves rate R' and distortion D'.

c) Infinite

d) zero

e) $R(\Delta) = \sum_{n=1}^{N} \max\left\{\frac{1}{2}\log_2\left(\frac{\sigma_n^2}{\Delta^2}\right), 0\right\}$

$D(\Delta) = \sum_{n=1}^{N} \min\left\{\sigma_n^2, \Delta^2\right\}$

Name: ____________________

f). $R(\Delta) = \sum_{n=1}^{N} \max\left\{\frac{1}{2}\log_2\left(\frac{\Lambda_{nn}}{\Delta^2}\right), 0\right\}$

$D(\Delta) = \sum_{n=1}^{N} \min\left\{\Lambda_{nn}, \Delta^2\right\}$

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Problem 5.(20pt)

Let $x(t) = \text{sinc}(at)$ for some positive constant a, and let $y(n) = x(nT)$ where $f_s = 1/T$ is the sampling frequency of the system. Further assume that a has units of sec^{-1}, T has units of sec, and f_s has units of $Hz = sec^{-1}$.

a) Calculate $X(f)$, the CTFT of $x(t)$.

b) What is the cutoff frequency of $X(f)$ in Hz? (Your answer will be in terms of a.)

c) Calculate $Y(e^{j\omega})$, the DTFT of $y(n)$.

d) Calculate f_q, the Nyquist sampling frequency for the signal. (Your answer will be in terms of a.)

e) Sketch the function $Y(e^{j\omega})$ on the interval $[-\pi, \pi]$ when $f_s = (3/2)f_q$.

f) Sketch the function $Y(e^{j\omega})$ on the interval $[-\pi, \pi]$ when $f_s = (2/3)f_q$.

a) $X(f) = \frac{1}{a}\text{rect}(f/a)$

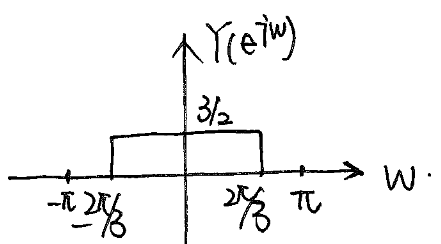
b) $\frac{a}{2}$

c) $Y(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{w-2\pi k}{2\pi T}\right)$

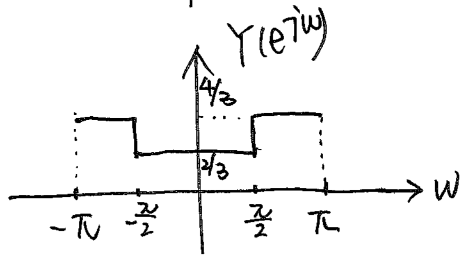
$= \frac{1}{aT}\sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{w-2\pi k}{2\pi aT}\right)$

d) $f_q = a$

e)



f)



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