

Image Restoration

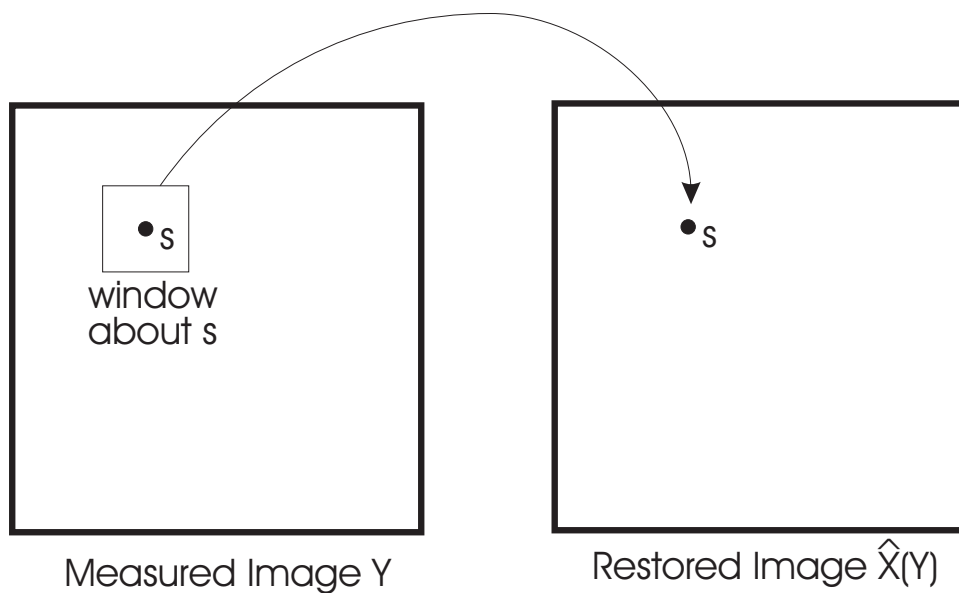
- Problem:
 - You want to know some image X .
 - But you only have a corrupted version Y .
 - How do you determine X from Y ?
- Corruption may result from:
 - Additive noise
 - Nonadditive noise
 - Linear distortion
 - Nonlinear distortion

Optimum Linear FIR Filter

- Find an “optimum” linear filter to compute X from Y .
- Filter uses input window of Y to estimate each output pixel X_s .
- Filter can be designed to be minimize mean squared error (MMSE).
- The estimate of X_s is denoted by $\hat{X}_s$.
- $W(s)$ denotes the window about s .
- The estimate, $\hat{X}_s$, is a function of $Y_{W(s)}$.

Application of Optimum Filter

$$\hat{X}_s = f(Y_{W(s)})$$



- The function $f(Y_{W(s)})$ is designed to produce a MMSE estimate of X .
- If $f(Y_{W(s)})$ is:
 - Linear $\Rightarrow$ linear space invariant filter.
 - Nonlinear $\Rightarrow$ nonlinear space invariant filter.
- This filter can reduce the effects of all types of corruption.

Optimality Properties of Linear Filter

- If both images are jointly Gaussian:
 - Then MMSE filter is linear.

$$\begin{aligned}\hat{X}_s &= E[X_s | Y_{W(s)}] \\ &= \mathbf{A} Y_{W(s)} + b\end{aligned}$$

- If images are not jointly Gaussian:
 - Then MMSE filter is generally not linear.

$$\begin{aligned}\hat{X}_s &= E[X_s | Y_{W(s)}] \\ &= f(Y_{W(s)})\end{aligned}$$

- However, the MMSE linear filter can still be very effective!

Formulation of MMSE Linear Filter: Definitions

- $W(s)$ - window about the pixel s .
- p - number of pixels in $W(s)$
- z_s - row vector containing pixels of $Y_{W(s)}$.
- θ - column containing filter parameters
- Detailed definitions:

- Definition of $W(s)$

$$W(s) = [s, s + r_1, \dots, s + r_{p-1}]$$

where $r_1, \dots, r_{p-1}$ index neighbors.

- Definition of z_s

$$z_s = [y_s, y_{s+r_1}, \dots, y_{s+r_{p-1}}]$$

- Definition of θ

$$\theta = [\theta_0, \dots, \theta_{p-1}]$$

Formulation of MMSE Linear Filter: Objectives

- Linear filter is given by

$$\hat{x}_s = z_s \theta$$

- Mean squared error is given by

$$\begin{aligned} MSE &= E[|x_s - \hat{x}_s|^2] \\ &= E[|x_s - z_s \theta|^2] \end{aligned}$$

- The MMSE filter parameters θ^* are given by

$$\theta^* = \arg \min_{\theta} E[|x_s - z_s \theta|^2] .$$

- How do we solve this problem?

More Matrix Notation

- Define the subset S_0 of image pixels.
 1. $S_0 \subset S$
 2. S_0 contains $N_0 < N$ pixels
 3. S_0 usually does not contain pixels on the boundary of the image.
 4. $S_0 = [s_1, \dots, s_{N_0}]$
- Define the $N_0 \times p$ matrix Z

$$Z = \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} .$$

- Define the $N_0 \times 1$ column vectors X and $\hat{X}$

$$X = \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{bmatrix} \hat{x}_{s_1} \\ \hat{x}_{s_2} \\ \vdots \\ \hat{x}_{s_{N_0}} \end{bmatrix} .$$

- Then

$$X \approx \hat{X} = Z\theta$$

Least Squares Linear Filter

- We expect that

$$\begin{aligned}MSE &= E[|x_s - z_s\theta|^2] \\&\approx \frac{1}{N_0} \sum_{s \in S_0} |x_s - z_s\theta|^2 \\&= \frac{1}{N_0} \|X - Z\theta\|^2\end{aligned}$$

- So we may solve the equation

$$\theta^* = \arg \min_{\theta} \|X - Z\theta\|^2$$

- The solution θ^* is the least squares estimate, of θ , and the estimate

$$\hat{X} = Z\theta^*$$

is known as the least squares filter.

Computing Least Squares Linear Filter

$$\theta^* = \arg \min_{\theta} \frac{1}{N_0} \|X - Z\theta\|^2$$

• So

$$\begin{aligned} \theta^* &= \arg \min_{\theta} \left(\frac{1}{N_0} \|X - Z\theta\|^2 \right) \\ &= \arg \min_{\theta} \left(\frac{1}{N_0} (X - Z\theta)^t (X - Z\theta) \right) \\ &= \arg \min_{\theta} \left(\frac{1}{N_0} (X^t X - 2\theta^t Z^t X + \theta^t Z^t Z \theta) \right) \\ &= \arg \min_{\theta} \left(\frac{X^t X}{N_0} - 2\theta^t \frac{Z^t X}{N_0} + \theta^t \frac{Z^t Z}{N_0} \theta \right) \\ &= \arg \min_{\theta} \left(\theta^t \frac{Z^t Z}{N_0} \theta - 2\theta^t \frac{Z^t X}{N_0} \right) \end{aligned}$$

Covariance Estimates

- Define the $p \times p$ matrix

$$\begin{aligned}
 \hat{R}_{zz} &\triangleq \frac{Z^t Z}{N_0} \\
 &= \frac{1}{N_0} \begin{bmatrix} z_{s_1}^t & z_{s_2}^t & \cdots & z_{s_{N_0}}^t \end{bmatrix} \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} \\
 &= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t z_{s_i}
 \end{aligned}$$

- Define the $p \times 1$ vector

$$\begin{aligned}
 \hat{r}_{zx} &\triangleq \frac{Z^t X}{N_0} \\
 &= \frac{1}{N_0} \begin{bmatrix} z_{s_1}^t & z_{s_2}^t & \cdots & z_{s_{N_0}}^t \end{bmatrix} \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \\
 &= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t x_{s_i}
 \end{aligned}$$

- So

$$\theta^* = \arg \min_{\theta} \left(\theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

Interpretation of $\hat{R}_{zz}$ and $\hat{r}_{zx}$

- $\hat{R}_{zz}$ is an estimate of the covariance of z_s .

$$\begin{aligned} E \left[\hat{R}_{zz} \right] &= E \left[\frac{1}{N_0} \sum_{s=1}^N z_s^t z_s \right] \\ &= E[z_s^t z_s] \\ &= R_{zz} \end{aligned}$$

- $\hat{r}_{zx}$ is an estimate of the cross correlation between z_s and x_s .

$$\begin{aligned} E \left[\hat{r}_{zx} \right] &= E \left[\frac{1}{N_0} \sum_{s=1}^N z_s^t x_s \right] \\ &= E[z_s^t x_s] \\ &= r_{zx} \end{aligned}$$

Solution to Least Squares Linear Filter

- We need

$$\theta^* = \arg \min_{\theta} \frac{1}{N_0} \|X - Z\theta\|^2$$

We have shown this is equivalent to

$$\theta^* = \arg \min_{\theta} \left(\theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

- Taking the gradient of the cost functional

$$\begin{aligned} 0 &= \nabla_{\theta} \left(\theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right) \Big|_{\theta=\theta^*} \\ &= \left(2\hat{R}_{zz} \theta - 2\hat{r}_{zx} \right) \Big|_{\theta=\theta^*} \end{aligned}$$

Solving for θ^* yeilds

$$\theta^* = \left(\hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

Summary of Solution to Least Squares Linear Filter

- First compute

$$\hat{R}_{zz} = \frac{1}{N_0} \sum_{s=1}^N z_s^t z_s$$

$$\hat{r}_{zx} = \frac{1}{N_0} \sum_{s=1}^N z_s^t x_s$$

- Then compute

$$\theta^* = \left(\hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

- The vector θ^* then contains the values of the filter coefficients.

Training

- θ^* is usually estimated from “training” data.
- Training data
 - Generally consists of image pairs (X, Y) where Y is the measured data and X is the undistorted image.
 - Should be typical of what you might expect.
 - Can often be difficult to obtain.
- Testing data
 - Also consists of image pairs (X, Y) .
 - Is used to evaluate the effectiveness of the filters.
 - Should never be taken from the training data set.
- Training versus Testing
 - Performance on training data is always better than performance on testing data.
 - As the amount of training data increases, the performance on training and testing data both approach the best achievable performance.

Comments

- Wiener filter is the MMSE **linear** filter.
- Wiener filter may be optimal, but it isn't always good.
 - Linear filters blur edges
 - Linear filters work poorly with non-Gaussian noise.
- Nonlinear filters can be designed using the same methodologies.

Is MMSE a Good Quality Criteria for Images?

- In general, NO! ... But sometimes it is OK.
- For achromatic images, it is best to choose X and Y in L^* or gamma corrected coordinates.
- Let H be a filter that implements the CSF for the human visual system.
 - Then a better metric of error is

$$\begin{aligned} HVSE &= \|H(X - \hat{X})\|^2 \\ &= (X - \hat{X})^t H^t H (X - \hat{X}) \\ &= \|X - \hat{X}\|_B^2 \end{aligned}$$

where $B = H^t H$.

- $\|X - \hat{X}\|_B^2$ is a quadratic norm.
- What is the minimum HVSE estimate $\hat{X}$?

Answer

- The answer is $\hat{X} = E[X|Y]$.
 - This is the same as for mean squared error!
 - The conditional expectation minimizes any quadratic norm of the error.
 - This is also true for non-Gaussian images.
- Let $\hat{X} = AY_{W(s)} + b$ be the MMSE **linear** filter.
 - This filter is also the minimum HVSE **linear** filter.
 - This is also true for non-Gaussian images.

Proof

- Define $V \triangleq HX$ and $B = H^t H$

$$\begin{aligned}
 & \min_{\hat{X}} E \left[||X - \hat{X}||_B^2 \right] \\
 &= \min_{\hat{X}} E \left[||H(X - \hat{X})||^2 \right] \\
 &= \min_{\hat{V}} E \left[||V - \hat{V}||^2 \right] \\
 &= E \left[||V - E[V|Y]||^2 \right] \\
 &= E \left[||HX - E[HX|Y]||^2 \right] \\
 &= E \left[||H(X - E[X|Y])||^2 \right] \\
 &= E \left[||X - E[X|Y]||_B^2 \right]
 \end{aligned}$$

- So, $\hat{X} = E[X|Y]$ minimizes the error measure.

$$HVSE = ||X - \hat{X}||_B^2 .$$