Topics: Discrete transforms; 1 and 2D Filters, sampling, and scanning

Spring 2010 Exam 2: Problem 3 (sampling)

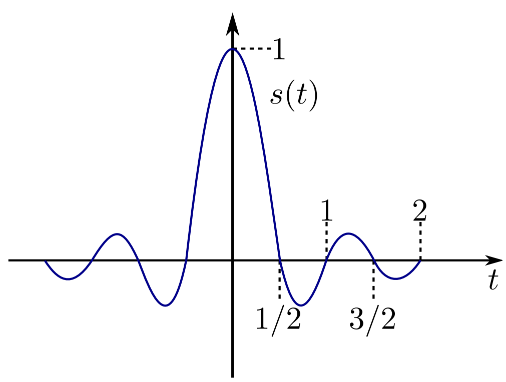
Consider a sampling system were the input, $s(t) = \text{sinc}(2t)$, is sampled with period $T = 1/2$ to form the sampled signal $x(n) = s(nT)$.

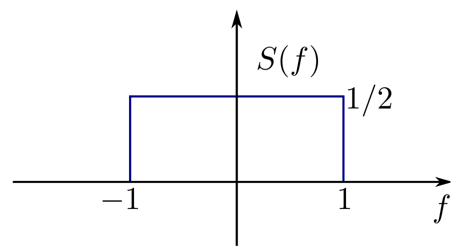
After sampling, you determine that you selected the wrong sampling rate, and really need to have sampled the signal at the period $T_2 = 1/4$; so you interpolate by a factor of $L = 2$ to form the signal $y(n)$.

a) Sketch the signal $s(t)$ and its CTFT $S(f)$. What is the Nyquist sampling rate for this signal?

Solution:

$$S(f) = \frac{1}{2} rect\left(\frac{t}{2}\right)$$

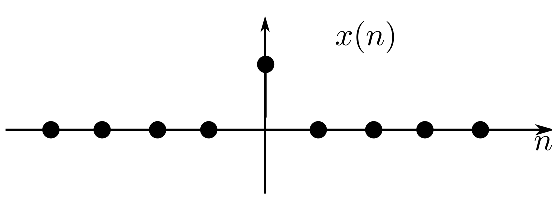


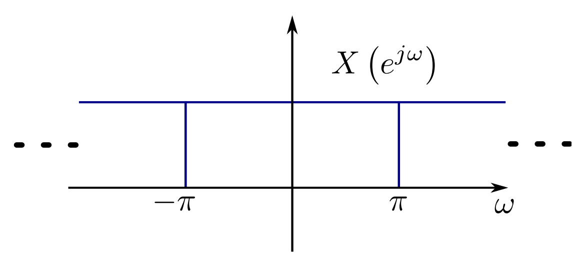


b) Sketch the signal $x(n)$ and also sketch its DTFT $X(e^{j\omega})$.

Solution:

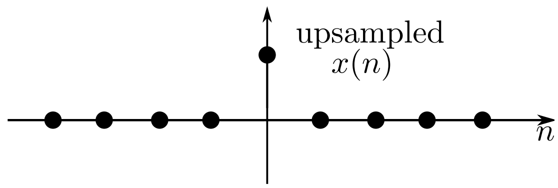
$$x(n) = s\left(\frac{1}{2}n\right) = \delta(n)$$





c) Sketch $x(n)$ after it is up-sampled by $L = 2$.

Solution:

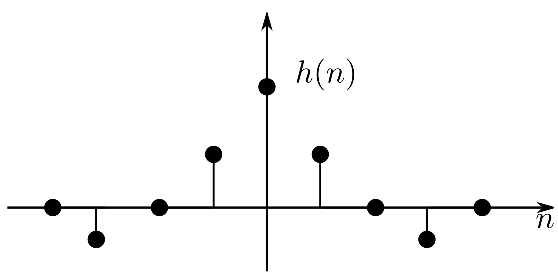


d) Sketch the interpolation filter's impulse response.

Solution:

$$h(n) = sinc\left(\frac{n}{L}\right)$$
$$H\left(e^{j\omega}\right) = Lrect\left(\frac{\omega}{\frac{2\pi}{L}}\right) \text{ , for } |\omega| < \pi$$

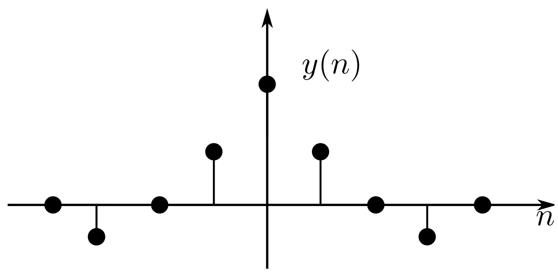
$$\text{So } h(n) = sinc\left(\frac{n}{2}\right)$$



e) Sketch the signal $y(n)$.

Solution:

$$\begin{aligned} y(n) &= \delta(n) * sinc\left(\frac{n}{2}\right) \\ &= sinc\left(\frac{n}{2}\right) \end{aligned}$$



f) What is the relationship between $y(n)$ and $s(t)$?

Solution:

When $T_2 = \frac{1}{4}$:

$$\begin{aligned} y(n) &= s(nT_2) \\ &= sinc\left(2\left(n\frac{1}{4}\right)\right) \\ &= sinc\left(\frac{n}{2}\right) \end{aligned}$$

Spring 2009 Exam 1: Problem 1 (FIR filters and frequency response)

Consider the linear-space invariant FIR filter given by

$$y(m,n) = x(m,n) * h(m,n)$$

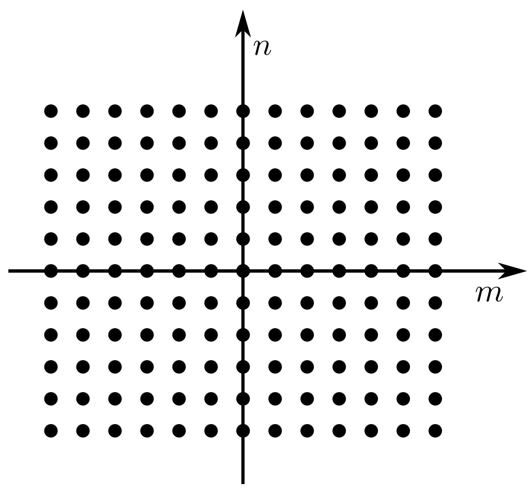
where

$$h(m,n) = \begin{cases} \pi & \text{if } |m| \le 5 \text{ and } |n| \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the function $h(m,n)$. You may use any method you prefer to sketch it (i.e. 2D or 3D sketch), but make sure to clearly show the zero and nonzero values and their locations in the plane.

Solution:

All values in the discrete window will be π. All other values are 0.



b) Calculate $H(e^0, e^0)$, the DC gain of the FIR filter.

Solution:

$$\begin{aligned} H\left(e^0, e^0\right) &= \sum_{m=-5}^{5} \sum_{n=-5}^{5} h(m,n) e^{j2\pi(0m+0n)} \\ &= \sum_{m=-5}^{5} \sum_{n=-5}^{5} h(m,n) \\ &= 121\pi \end{aligned}$$

c) Is this function separable? If so, then give its separable decomposition $h(m,n) = g(m)f(n)$.

Solution:

Yes, $h(m,n) = g(m)f(n)$ where

$$g(m) = \begin{cases} \sqrt{\pi} & \text{if } |m| \le 5 \\ 0 & \text{otherwise} \end{cases} \text{ and } f(n) = \begin{cases} \sqrt{\pi} & \text{if } |n| \le 5 \\ 0 & \text{otherwise} \end{cases}$$

d) How many multiplies per output point are required for direct implementation of the FIR filter output?

Solution:

Direct implementation of this filter would require 121 multiplies.

e) Specify an alternative implementation which uses the separable nature of the FIR filter.

Solution:

Instead of using $h(m,n)$, first filter using $g(m)$, then filter using $f(n)$.

f) How many multiplies per output point are required for separable implementation of the FIR filter output?

Solution:

This alternate method would only require 10 multiplies per output (five from each filter).

Spring 2007 Exam 1: Problem 1 (DSFT and 2D Z-transform)

Consider the following 2D system with input $x(m,n)$ and output $y(m,n)$.

$$y(m,n) = x(m,n) + \lambda\left(x(m,n) - \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} x(m-k, n-l)\right) .$$

a) Is this a linear system? Is this a space invariant system?

Solution:

Yes, it is a linear space invariant system.

b) What is the 2D impulse response of this system, $h(m,n)$?

Solution:

$$\begin{aligned} h(m,n) &= \delta(m,n)(1+\lambda) - \frac{\lambda}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1}\delta(m-k,n-l) \\ &= \begin{cases} \left(1+\frac{8\lambda}{9}\right) & \text{, for } m=n=0 \\ \frac{-\lambda}{9} & \text{, for } |m|\le 1,\ |n|\le 1, \text{ but } m\neq 0 \text{ and } n\neq 0 \\ 0 & \text{, otherwise} \end{cases} \end{aligned}$$

c) Calculate the frequency response, $H(e^{j\mu}, e^{j\nu})$?

Solution:

$$\begin{aligned} H\left(e^{j\mu}, e^{j\nu}\right) &= (1+\lambda) - \frac{\lambda}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} e^{-j\mu k}e^{-j\nu l} \\ &= (1+\lambda) - \frac{\lambda}{9}\sum_{k=-1}^{1} e^{-j\mu k}\sum_{l=-1}^{1} e^{-j\nu l} \\ &= (1+\lambda) - \frac{\lambda}{9}\left(e^{-j\mu}+1+e^{j\mu}\right)\left(e^{-j\nu}+1+e^{j\nu}\right) \\ &= (1+\lambda) - \frac{\lambda}{9}\left(1+2\cos\mu\right)\left(1+2\cos\nu\right) \end{aligned}$$

d) Describe how the filter behaves when λ is positive and large.

Solution:

For $\lambda > 0$ and large, the filter performs sharpening.

e) Describe how the filter behaves when λ is negative and > -1.

Solution:

For $-1 < \lambda < 0$, the filter performs blurring.

Spring 2007 Exam 1: Problem 2 (DSFT and 2D Z-transform)

Consider the causal linear space invariant system with input $x(m,n)$ and output $y(m,n)$ that is specified by

$$y(m,n) = x(m,n) + ay(m-1,n) + by(m,n-1)$$

a) Calculate the transfer function $H(z_1, z_2)$ for this system.

Solution:

$$H(z_1, z_2) = \frac{1}{1 - az_1^{-1} - bz_2^{-1}}$$

b) Calculate the value of

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m,n)$$

where $h(m,n)$ is the 2D impulse response of the system.

Solution:

$$H\left(e^{j\mu}, e^{j\nu}\right) = \frac{1}{1 - ae^{j\mu} - be^{j\nu}}$$

$$\sum_m \sum_n h(m,n) = H\left(e^{j0}, e^{j0}\right) = \frac{1}{1-a-b}$$

c) Calculate the value of

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m,n)\cos(\omega_o m) \ .$$

Solution:

$$\text{Let } \tilde{h}(m,n) = h(m,n)cos(\omega_0 m)$$

$$\widetilde{H}\left(e^{j\mu}, e^{j\nu}\right) = \frac{1}{2} H\left(e^{j(\mu-\omega_0)}, e^{j\nu}\right) + \frac{1}{2} H\left(e^{j(\mu+\omega_0)}, e^{j\nu}\right)$$

$$\begin{aligned}
\sum_m \sum_n \tilde{h}(m,n) &= \widetilde{H}\left(e^{j0}, e^{j0}\right) \\
&= \frac{1}{2} H\left(e^{-j\omega_0}, e^{j0}\right) + \frac{1}{2} H\left(e^{j\omega_0}, e^{j0}\right) \\
&= \left(\frac{1}{2}\right) \frac{1}{1 - ae^{jw_0} - b} + \left(\frac{1}{2}\right) \frac{1}{1 - ae^{-jw_0} - b}
\end{aligned}$$

d) Is this system stable for all, none, or some values of (a,b)? Justify your answer.

Solution:

The system will be stable for some values of a and b. For instance, it will be unstable for $a = b = 0.5$ and stable for $a = b = 0.2$.