Topics: Continuous 1 and 2D Fourier Transform

Spring 2009 Final: Problem 1 (CSFT and DTFT properties)

Derive each of the following properties.

a) Show that if $g(t)$ has a CTFT of $G(f)$, then $g(t-a)$ has a CTFT of $e^{-2\pi jaf}G(f)$.

Solution:

Let $\mathcal{F}\{\cdot\}$ denote the Fourier transform operator.

$$\begin{aligned}
\mathcal{F}\{g(t-a)\} &= \int_{-\infty}^{\infty} g(t-a)e^{-j2\pi ft}dt, \text{ let } t' = t-a \\
&= \int_{-\infty}^{\infty} g(t')e^{-j2\pi f(t'+a)}dt' \\
&= e^{-j2\pi fa}\int_{-\infty}^{\infty} g(t')e^{j2\pi ft'}dt' \\
&= e^{-j2\pi fa}G(t)
\end{aligned}$$

b) Show that if $g(t)$ has a CTFT of $G(f)$, then $g(t/a)$ has a CTFT of $|a|G(af)$.

$$\mathcal{F}\left\{g\left(\frac{t}{a}\right)\right\} = \int_{-\infty}^{\infty} g\left(\frac{t}{a}\right)e^{-j2\pi ft}dt$$

Solution:

1) When $a > 0$, $t' = \frac{t}{a}$:

$$\begin{aligned}
\mathcal{F}\left\{g\left(\frac{t}{a}\right)\right\} &= \int_{-\infty}^{\infty} g(t')e^{-j2\pi fat'}adt' \\
&= a\int_{-\infty}^{\infty} g(t')e^{-j2\pi fat'}dt' \\
&= aG(af)
\end{aligned}$$

2) When $a < 0$, $t' = \frac{t}{a}$:

$$\begin{aligned}
\mathcal{F}\left\{g\left(\frac{t}{a}\right)\right\} &= \int_{\infty}^{-\infty} g(t')e^{-j2\pi fat'}adt' \\
&= -a\int_{-\infty}^{\infty} g(t')e^{-j2\pi fat'}dt' \\
&= -aG(af)
\end{aligned}$$

So, $\mathcal{F}\left\{g\left(\frac{t}{a}\right)\right\} = |a|G(af)$.

c) Show that if x_n has a DTFT of $X(e^{j\omega})$, then $(-1)^n x_n$ has a DTFT of $X(e^{j(\omega-\pi)})$.

Solution:

$$\begin{aligned}
\text{DTFT}\left\{(-1)^n x_n\right\} &= \sum_{n=-\infty}^{\infty} (-1)^n x_n e^{-jwn} \\
&= \sum_{n=-\infty}^{\infty} e^{j\pi n} x_n e^{-jwn} \\
&= \sum_{n=-\infty}^{\infty} x_n e^{-j(w-pi)n} \\
&= X\left(e^{j(w-\pi)}\right)
\end{aligned}$$

d) Show that if $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ has a CSFT of $G\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$, then $g\left(A\begin{bmatrix} x \\ y \end{bmatrix}\right)$ has a CSFT of $|A|^{-1}G\left(\left(A^{-1}\right)^t\begin{bmatrix} u \\ v \end{bmatrix}\right)$.

(Hint: Use the notation $r = \begin{bmatrix} x \\ y \end{bmatrix}$ and $f = \begin{bmatrix} u \\ v \end{bmatrix}$, so that $G(f) = \int_{\Re^2} g(r)e^{-jr^t f}dr$.)

Solution:

$$\begin{aligned}
\mathcal{F}\left\{g(Ar)\right\} &= \int_{\mathbb{R}^2} g(Ar)e^{-jr^t f}dr,\ r' = Ar,\ r = A^{-1}r' \\
&= \int_{\mathbb{R}^2} g(r')e^{-j(A^{-1}r')^t f}|A^{-1}|dr,\ \text{the Jacobian matrix for this transform is } A^{-1} \\
&= \int_{\mathbb{R}^2} g(r')e^{-j(A^{-1}r')^t f}|A|^{-1}dr,\ \text{because } |A^{-1}| = |A|^{-1} \\
&= |A|^{-1}\int_{\mathbb{R}^2} g(r')e^{-j(r')^t(A^{-1})^t f}dr \\
&= |A|^{-1}G\left((A^{-1})^t f\right)
\end{aligned}$$

Spring 2008 Exam 1: Problem 1 (CSFT)

Consider the CSFT given by

$$F(u,v) = \frac{1}{2}\left[\delta(u-u_o, v-v_o) + \delta(u+u_o, v+v_o)\right]$$

a) Calculate, $f(x,y)$, the inverse CSFT of $F(u,v)$.

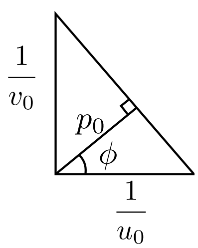
Solution:

Using the identity $e^{j2\pi(u_0x+v_0y)} \overset{CSFT}{\Longleftrightarrow} \delta(u-u_o, v-v_o)$, we have:

$$\begin{aligned} f(x,y) &= \frac{1}{2}\left(e^{j2\pi(u_0x+v_0y)} + e^{-j2\pi(u_0x+v_0y)}\right) \\ &= \cos\left(2\pi(u_0x+v_0y)\right) \end{aligned}$$

b) Calculate the minimum distance between nearest peaks in the function $f(x,y)$.

Solution:



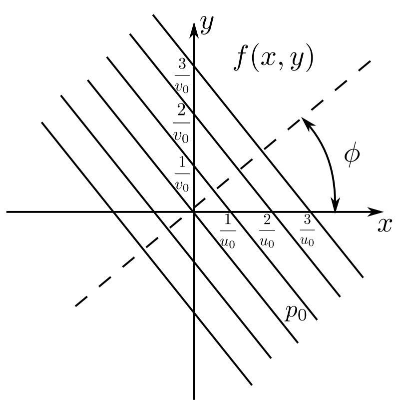
$$\phi = \arctan\frac{v_0}{u_0} \implies \cos\phi = \frac{u_0}{\sqrt{u_0^2+v_0^2}}, \text{ and } \sin\phi = \frac{v_0}{\sqrt{u_0^2+v_0^2}}$$

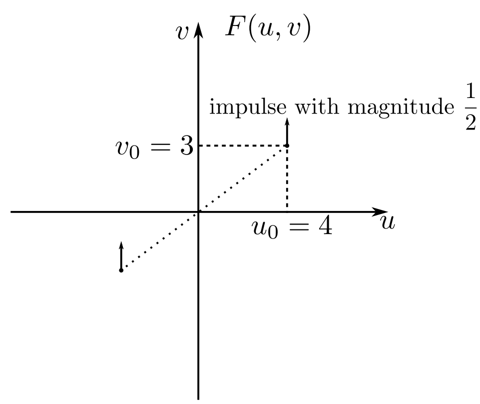
$$\therefore p_0 = \frac{1}{u_0}\cos\phi = \frac{1}{v_0}\sin\phi = \frac{1}{\sqrt{u_0^2+v_0^2}}$$

i.e., the minimum distance between nearest peaks is $\dfrac{1}{\sqrt{u_0^2+v_0^2}}$

c) Sketch $F(u,v)$ and $f(x,y)$ when $u_o = 4$ and $v_o = 3$. Label the axis on your sketch, and also make sure to label important dimensions of the signal and its transform.

Solution:





The lines of the plot of $f(x,y)$ represent the locations of the 2D plane wave peaks, which have a magnitude of 1.

Spring 2006 Exam 1: Problem 2 (CSFT)

a) Calculate the CSFT of

$$f(x,y) = rect\left(x/A, x/B\right)$$

Solution:

From the provided "Fact Sheet" and using separability, we have

$$F(u,v) = |AB| sinc(uA, vB)$$

b) Calculate the CSFT of

$$g(x,y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f\left(x - 5kA, y - 5lB\right)$$

Solution:

$$\begin{aligned} g(x,y) &= rep_{5A,5B}\left\{f(x,y)\right\} \\ G(u,v) &= \frac{1}{25}\frac{1}{|AB|} comb_{\frac{1}{5A},\frac{1}{5B}}\left\{|AB| sinc(uA, vB)\right\} \\ &= \frac{1}{25} comb_{\frac{1}{5A},\frac{1}{5B}}\left\{sinc(uA, vB)\right\} \end{aligned}$$

c) Calculate the CSFT of

$$h(x,y) = \begin{cases} g(x,y) & \text{for } |x| < T/2 \text{ and } |y| < T/2 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} h(x,y) &= rect\left(\frac{x}{T}, \frac{y}{T}\right) g(x,y) \\ H(u,v) &= T^2 sinc(uT, vT) * G(u, V) \end{aligned}$$

d) Sketch $h(x, y)$ for $A = B = 1$ and $T = 50$.

Solution: