EE 637 Final

May 7, Spring 2014

**Name:** ______________________________

**Instructions:**

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

## Good Luck.

# Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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**Problem 1.**(20pt)

Consider the continuous space signal $g(x, y)$ with corresponding CSFT $G(u, v)$, and the sampled signal $f(m, n) = g(Tx, Ty)$ with corresponding DSFT $F(e^{j\mu}, e^{j\nu})$.

a) Write a formula for $F(e^{j\mu}, e^{j\nu})$ in terms of the function $G(u, v)$.

b) Give sufficient conditions for perfect reconstruction of $g(x, y)$ from $f(m, n)$.

c) Derive a discrete approximation for the gradient, $\nabla g(x, y)$, as a function of $f(m, n)$.

d) Derive a discrete approximation for the Laplacian, $\nabla^2 g(x, y)$, as a function of $f(m, n)$.

e) Use the two expressions of parts c) and d) above to specify a useful edge detection algorithm.

a) $$F(e^{j\mu}, e^{j\nu}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right)$$

b) In order to meet the Nyquist criteria, it is necessary that sampling rate is greater than or equal to the highest spatial frequency. So the frequencies must be limited to

$$\max\{|u|, |v|\} \leq \frac{1}{2T}$$

c) $$\nabla g(x, y) \approx Df(m, n) = \left[\frac{f(m+1, n) - f(m-1, n)}{2T}, \frac{f(m, n+1) - f(m, n-1)}{2T}\right]$$

d) $$\nabla^2 g(x, y) \approx D_2 f(m, n) = \frac{-4}{T^2}\left\{ f(m, n) - \frac{1}{4}\left[f(m+1, n) + f(m, n+1) + f(m-1, n) + f(m, n-1)\right]\right\}$$

e). A typical edge detection algorithm checks that the magnitude of the gradient is ~~below~~ above a threshod and the Laplacian is at a zero crossing

$$|Df(m, n)| \geq T$$

and $$[(D_2 f(m, n) \cdot D_2 f(m-1, n) < 0) \text{ or } (D_2 f(m, n) \cdot D_2 f(m, n-1) < 0)]$$

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**Problem 2.**(20pt)

Consider a 2D zero mean Gaussian random vector $X = [X_1, X_2]^t$ with $E[X_1^2] = E[X_2^2] = 1$. Furthermore, let $\rho$ denote the correlation between $X_1$ and $X_2$, so that $E[X_1 X_2] = \rho$.

a) Sketch a contour plot given by $p(x) = constant$ when $\rho = 0.9$. Label all relevant quantities on the sketch.

b) Let $R = E[XX^t]$ be the covariance matrix for the vector $X$. Determine the **eigen vectors** of $R$.

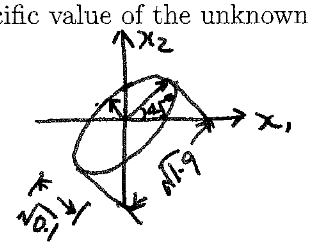
c) Let $R = E[XX^t]$ be the covariance matrix for the vector $X$. Determine the **eigen values** of $R$.

d) Imagine that you have two zero mean Gaussian random variables, $Y_1$ and $Y_2$ with variance $\sigma_1^2$ and $\sigma_2^2$ respectively, but with unknown correlation $\rho$. Furthermore, let $T$ be a linear transform so that

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = T \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

Find a fixed transform, $T$, such that the variables, $[Z_1, Z_2]$ are always decorrelated regardless of the specific value of the unknown correlation $\rho$.

a).




b.c). $R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ eigen values: $\det(R - \lambda I) = 0$

$\lambda = 1 \pm \rho$

Eigenvectors:

Case 1: $\lambda = 1 + \rho$

$R e_1 = \lambda e_1$

$e_1 = [1/\sqrt{2}, 1/\sqrt{2}]^t$

Case 2: $\lambda = 1 - \rho$

$R e_2 = \lambda e_2$

$e_2 = [1/\sqrt{2}, -1/\sqrt{2}]^t$

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d). How to choose $T$:

step 1. scale to get unit variance

step 2. $y_1+y_2$ and $y_1-y_2$ are decorrelated

For step 1: $T_1 = \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix}$

Quick proof: $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T_1 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \sigma_1^{-1} y_1 \\ \sigma_2^{-1} y_2 \end{bmatrix}$

$$E[z_1^2] = \sigma_1^{-2} E[Y_1^2] = 1$$

similarly $E[z_2^2] = 1$

For step 2:

$$T_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Together $T = T_2 T_1 = \begin{bmatrix} \sigma_1^{-1} & \sigma_2^{-1} \\ \sigma_1^{-1} & -\sigma_2^{-1} \end{bmatrix}$

proof: $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \sigma_1^{-1} Y_1 + \sigma_2^{-1} Y_2 \\ \sigma_1^{-1} Y_1 - \sigma_2^{-1} Y_2 \end{bmatrix}$

$$E[z_1 z_2] = E[z_2 z_1] = \sigma_1^{-2} E[Y_1^2] - \sigma_2^{-2} E[Y_2^2]$$
$$+ \sigma_1^{-1}\sigma_2^{-1} E[Y_1 Y_2] - \sigma_1^{-1}\sigma_2^{-1} E[Y_1 Y_2]$$
$$= \sigma_1^{-2}\sigma_1^{2} - \sigma_2^{-2}\sigma_2^{2}$$
$$= 1 - 1 = 0$$

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**Problem 3.**(20pt)

Consider the set of data $\{x_n\}_{n=0}^{N-1}$ for $N$ odd. We would like to estimate a "central value" using a method known as M-estimation. To do this we compute the following function

$$\hat{\theta} = \arg\min_{\theta} \left\{ \sum_{n=0}^{N-1} \rho(x_n - \theta) \right\}$$

where $\rho$ is a function with the properties that $\rho(\Delta) \geq 0$ and $\rho(-\Delta) = \rho(\Delta)$.

a) Compute a closed form expression for the M-estimator when $\rho(\Delta) = |\Delta|^2$.

b) What function $\rho(\Delta)$ will result in the M-estimator being the median?

c) Derive a nonlinear equation that can be solved for $\hat{\theta}$. Hint: Do this by differentiating the cost function that is being minimized in the M-estimator.

d) Give a specific example of a function $\rho(\Delta)$ so that there is no influence on $\hat{\theta}$ when $|\Delta| = |\hat{\theta} - x_k| > T$.

a) $\hat{\theta} = \arg\min_{\theta} \left\{ \sum_{n=1}^{N-1} (x_n - \theta)^2 \right\}$

$\frac{d}{d\theta} \sum_{n=0}^{N-1} (x_n - \hat{\theta})^2 = 0$

$2 \sum_{n=0}^{N-1} (x_n - \hat{\theta})(-1) = 0$

i.e. $N\hat{\theta} = \sum_{n=0}^{N-1} x_n$

$\therefore \hat{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} x_n$ ← mean value

$\frac{d}{d\theta^2} \sum_{n=0}^{N-1} (x_n - \theta)^2 = 2N > 0$

b) $\rho(\Delta) = |\Delta|$

c) $\hat{\theta} = \arg\min_{\theta} \left\{ \sum_{n=0}^{N-1} \rho(x_n - \theta) \right\}$

$\frac{d}{d\theta} \sum_{n=0}^{N-1} \rho(x_n - \hat{\theta}) = 0$ ← $1^{st}$ order condition

$\Rightarrow \sum_{n=0}^{N-1} \rho'(x_n - \hat{\theta}) = 0$

second order condition.

$\sum_{n=0}^{N-1} \rho''(x_n - \hat{\theta}) > 0$

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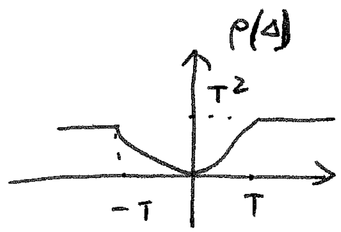
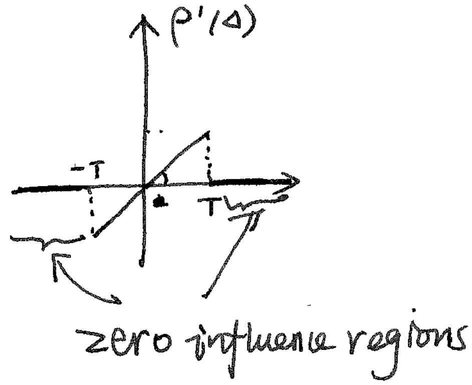
d) $\rho(\Delta) = \begin{cases} \Delta^2 & |\Delta| \le T \\ T^2 & |\Delta| > T \end{cases}$



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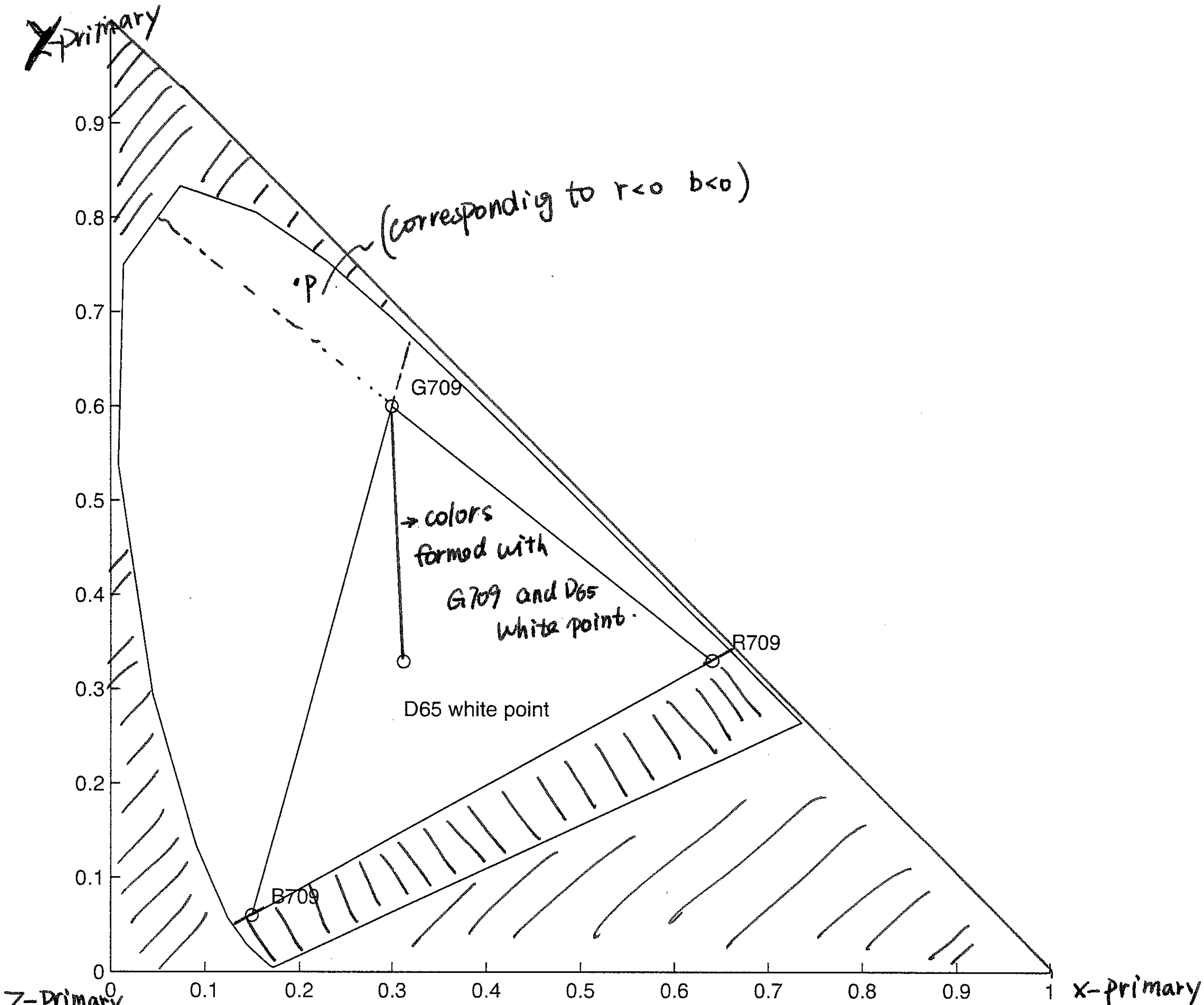
**Problem 4.**(20pt)

Consider the standard chromaticity diagram below, and assume that you are using a display device with standard 709 $r, g, b$ color primaries.




a) Draw a triangle corresponding to all colors with positive values of $X$, $Y$, and $Z$. Label the three primaries for this triangle as "$X$-primary", "$Y$-primary", and "$Z$-primary".

b) Label the region of the chromaticity diagram corresponding to imaginary colors. (Use 45 deg diagonal hash marks to indicate this region of the diagram.)

c) Label ALL real colors with $g < 0$ on the chromaticity diagram. (Use -45 deg diagonal hash marks to indicate this region of the diagram.)

d) Draw a line on the plot corresponding to all colors that can be formed with a combination of the D65 white and G709.

e) Place a point on the diagram corresponding to a single color with values of $b$ and $r$ that are both negative, i.e., $b < 0$ and $r < 0$. Label this point with the letter "P".

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**Problem 5.**(20pt)

Consider an MRI that only images in one dimension, $x$. So for example, the object being imaged might be a thin rod, $a(x)$, oriented along the x-dimension.

In this example, assume that the magnetic field strength at each location is given by

$$M_o + G_x(t)x$$

where $M_o$ is the static magnetic field strength and $G_x(t)$ is the linear gradient field in the $x$ dimension. Then the frequency of precession for a hydrogen atom (in rad/sec) is given by the product of $\gamma$, the gyromagnetic constant, and the magnetic field strength.

a) Calculate $\omega(x,t)$, the frequency of precession of a hydrogen atom at location $x$ and time $t$.

b) Calculate $\phi(x,t)$, the phase of precession of a hydrogen atom at location $x$ and time $t$ assuming that $\phi(x,0) = 0$.

c) Calculate $r(x,t)$, the signal radiated from hydrogen atoms in the interval $[x, x+dx]$ at time $t$ in terms of a(x), the quantity of precessing hydrogen atoms along the thin rod.

d) Calculate $r(t)$, the signal radiated from hydrogen atoms along the entire object.

e) Design a function, $G_x(t)$, so that it is possible to reconstruct $a(x)$ from $r(t)$. Discuss the rationale behind your selection, and explain how you can then reconstruct $a(x)$ from $r(t)$.

a) $\omega(x,t) = \gamma(M_o + G_x(t)x)$

b) $\frac{d}{dt}\phi(x,t) = \omega(x,t) = \gamma(M_o + G_x(t)x)$.

$$\phi(x,t) = \gamma M_o t + x\int_0^t \gamma G_x(\tau)d\tau$$
$$= \gamma M_o t + x K_x(t)$$

where $K_x(t) = \int_0^t \gamma G_x(\tau)d\tau$

c) $r(x,t) = a(x)\, e^{j\phi(x,t)} = a(x)\, e^{j\gamma M_o t} e^{jx K_x(t)}$

d) $r(t) = \int_{-\infty}^{\infty} r(x,t)\,dx = e^{j\gamma M_o t}\int_{-\infty}^{\infty} a(x)\, e^{-jx K_x(t)}\,dx$

e) Not that $r(t) = e^{j\gamma M_o t} A(-K_x(t))$

where $A(\omega) =$ CTFT$\{a(x)\}$

Therefore $a(x) =$ CTFT$^{-1}\{A(\omega)\}$

$$= \text{CTFT}^{-1}\left\{\frac{r(t)}{e^{jM_o t\gamma}}\right\}$$

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$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{r(t)}{e^{jM_0 t r}} e^{-j \underbrace{kx(t)}_{-\omega} x} \, dw \qquad \left(\text{where } w = -kx(t) \right.$$
$$\left. = -\int_0^t \gamma G_x(\tau)d\tau\right)$$

$$= \frac{\gamma(t)}{2\pi e^{jM_0 t r}} \int_{-\infty}^{\infty} e^{jwx} \, dw$$

$$G_x(t) = \begin{cases} G & t \le T \\ -G & T < t \le 2T \end{cases}$$

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