EE 637 Midterm I

February 21, Spring 2014

Name: (4 pt) ______________________________________

Instructions:

- This is a 50 minute exam containing **three** problems.
- You may **only** use your brain and a pencil (or pen) and the included "Fact Sheet" to complete this exam.
- You **may not** use or have access to your book, notes, any supplementary reference, a calculator, or any communication device including a cell-phone or computer.
- You **may not** communicate with any person other than the official proctor during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$
$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$
$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$
$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$
$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$
$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$
$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$
$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$
$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy$$
$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} du dv$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\text{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\text{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\text{comb}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}} [X(f)]$$

$$\text{rep}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$
$$Y(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T} \right)$$
$$s(t) = \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT)$$
$$S(e^{j\omega}) = Y\left(e^{j\omega T} \right)$$

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Problem 1.(32pt)

Consider the following 2-D LSI systems. The first system (S1) has input $x(m,n)$ and output $y(m,n)$, and the second system (S2) has input $y(m,n)$ and output $z(m,n)$.

$$y(m,n) = ay(m,n-1) + x(m,n) \qquad \text{(S1)}$$
$$z(m,n) = bz(m-1,n) + y(m,n) \qquad \text{(S2)}$$

The third system (S3) is formed by the composition of (S1) and (S2) with input $x(m,n)$ and output $z(m,n)$ and impulse response $h_3(m,n)$.

a) Calculate the 2-D impulse response, $h_1(m,n)$, of the first system (S1).

b) Calculate the 2-D impulse response, $h_2(m,n)$, of the second system (S2).

c) Calculate the 2-D impulse response, $h_3(m,n)$, of the complete system (S3).

d) Calculate the 2-D transfer function, $H_1(z_1, z_2)$, of the first system (S1).

e) Calculate the 2-D transfer function, $H_3(z_1, z_2)$, of the first system (S3).

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Problem 2.(35pt)

Consider a function, $f(x, y)$, with a CSFT of $F(u, v)$, and imagine that we have two pieces of information about the function.

- First, we know its projection at angle $\theta = 0$ is given by:

$$p_0(t) = \int_{-\infty}^{\infty} f(t, y)\, dy \ .$$

- Second, we know that it is rotationally invariant so that for all θ,

$$f(x, y) = f\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \ .$$

Furthermore, define the CTFT of the projection as

$$P_0(\rho) \ = \ \int_{-\infty}^{\infty} p_0(t) e^{-j2\pi\rho t} dt \ .$$

a) Derive an expression for $P_0(\rho)$ in terms of $F(u, v)$.

b) Derive an expression for $F(u, v)$ in terms of $P_0(\rho)$.

c) Is it possible to exactly reconstruct $f(x, y)$ from these two pieces of information? If so, then how? If not, then why not?

d) Derive an expression for $f(x, y)$ when $p_0(t) = \text{sinc}(t)$.

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Problem 3.(32pt)

Consider the discrete-time LTI system with input $x(n)$ and output $y(n)$ given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) \ ,$$

where

$$h(n) = T\mathrm{sinc}(nT) \ .$$

a) Calculate a closed form expression for the frequency response, $H(e^{j\omega})$, of the LTI system. (Hint: You can use the sampling formula.)

b) Sketch the function, $H(e^{j\omega})$, on the interval $[-2\pi, 2\pi]$ for $T = 1/2$.
Also sketch it for $T = 1$ and $T = 3/2$.

c) Sketch the impulse response, $h(n)$, on the interval $n = -5, \cdots, +5$ for $T = 1/2$.
Also sketch it for $T = 1$ and $T = 3/2$.

d) If you would like to design a discrete-time filter with a cut-off frequency of ω_c, how should T be chosen?

e) Calculate the $\sum_{n=-\infty}^{\infty} h(n)$ when $|T| < 1$.

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