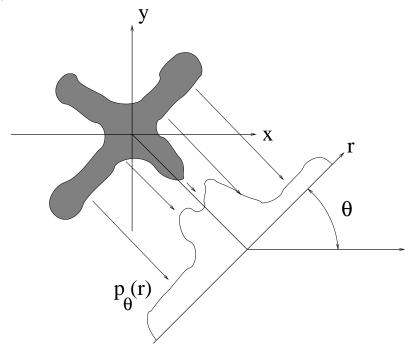
Tomography

- Many medical imaging systems can only measure projections through an object with density f(x, y).
 - Projections must be collected at every angle θ and displacement r.
 - Forward projections $p_{\theta}(r)$ are known as a Radon transform.



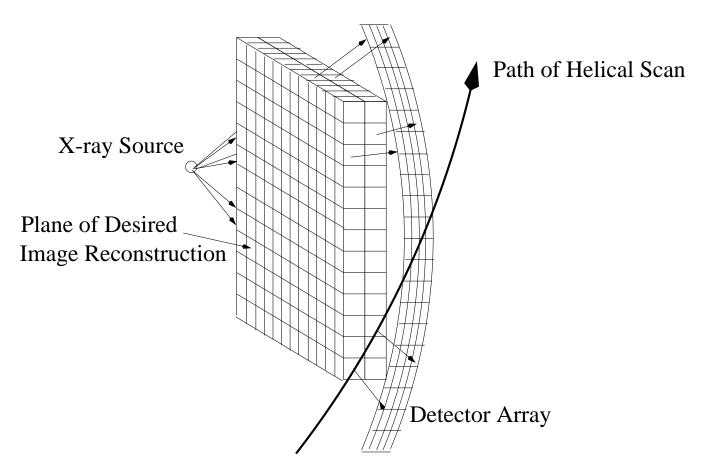
- ullet Objective: reverse this process to form the original image f(x,y).
 - Fourier Slice Theorem is the basis of inverse
 - Inverse can be computed using convolution back projection (CBP)

Medical Imaging Modalities

- Anatomical Imaging Modalities
 - Chest X-ray
 - Computed Tomography (CT)
 - Magnetic Resonance Imaging (MRI)
- Functional Imaging Modalities
 - Signal Photon Emission Tomography (SPECT)
 - Positron Emission Tomography (PET)
 - Functional Magnetic Resonance Imaging (fMRI)

Multislice Helical Scan CT

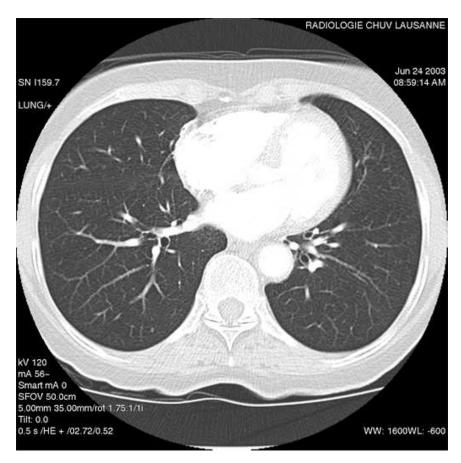
• Multislice CT has a cone-beam structure



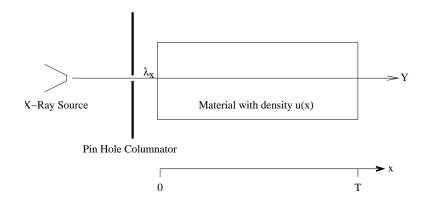
Example: CT Scan



- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan



Photon Attenuation



x - depth into material measured in cm

 Y_x - Number of photons at depth x

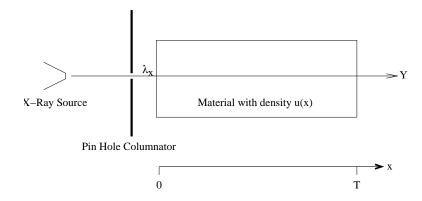
$$\lambda_x = E[Y_x]$$

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x} \lambda_x^k}{k!}.$$

- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.

Differential Equation for Photon Attenuation



The attenuation of photons obeys the following equation

$$\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x$$

where $\mu(x)$ is the density in units of cm⁻¹.

• The solution to this equation is given by

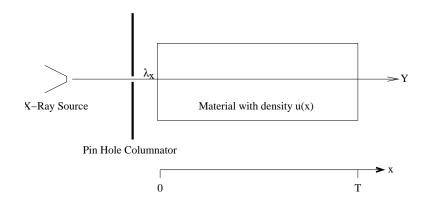
$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t)dt}$$

• So we see that

$$\int_0^x \mu(t)dt = -\log\left(\frac{\lambda_x}{\lambda_0}\right)$$

$$\approx -\log\left(\frac{Y_x}{\lambda_0}\right)$$

Estimate of the Projection Integral



A commonly used estimate of the projection integral is

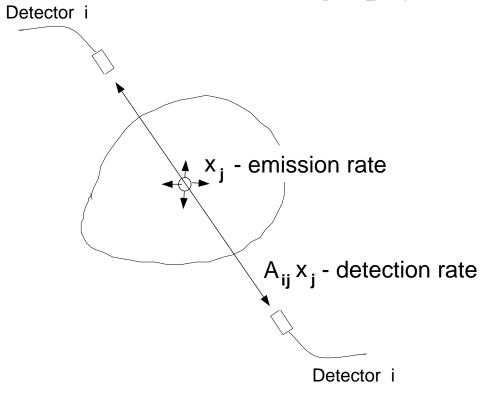
$$\int_0^T \mu(t)dt \stackrel{\sim}{=} -\log\left(\frac{Y_T}{\lambda_0}\right)$$

where:

 λ_0 is the dosage

 Y_T is the photon count at the detector

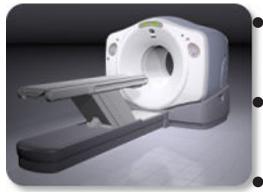
Positron Emission Tomography (PET)



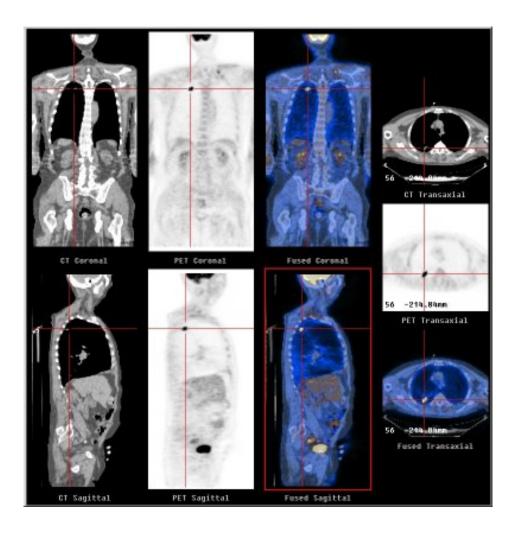
$$E[y_i] = \sum_j A_{ij} x_j$$

- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements

Example: PET/CT Scan



- Generally low space/time resolution
- Little anatomical detail ⇒ couple with CT
- Can detect disease



Coordinate Rotation

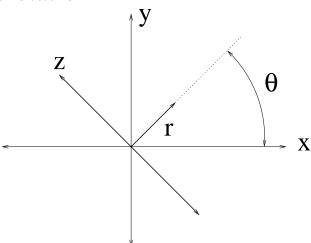
• Define the counter-clockwise rotation matrix

$$\mathbf{A}_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

ullet Define the new coordinate system (r, z)

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \mathbf{A}_{\theta} \left[\begin{array}{c} r \\ z \end{array}\right]$$

• Geometric interpretation

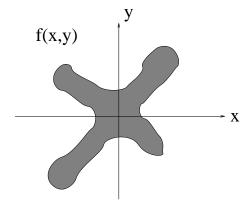


• Inverse transformation

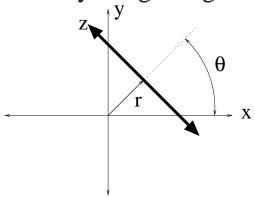
$$\left[\begin{array}{c} r \\ z \end{array}\right] = \mathbf{A}_{-\theta} \left[\begin{array}{c} x \\ y \end{array}\right]$$

Integration Along Projections

• Consider the function f(x, y).



ullet We compute projections by integrating along z for each r.



• The projection integral for each r and θ is given by

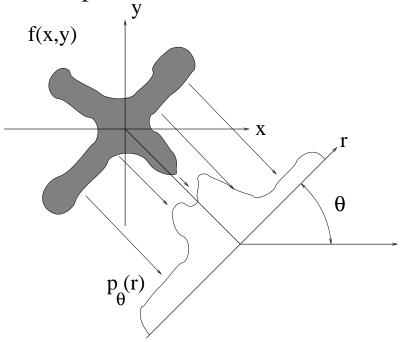
$$p_{\theta}(r) = \int_{-\infty}^{\infty} f\left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix}\right) dz$$
$$= \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz$$

The Radon Transform

ullet The Radon transform of the function f(x,y) is defined as

$$p_{\theta}(r) = \int_{-\infty}^{\infty} f(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)) dz$$

• The geometric interpretation is



Notice that the projection corresponding to r=0 goes through the point (x,y)=(0,0).

The Fourier Slice Theorem

• Let

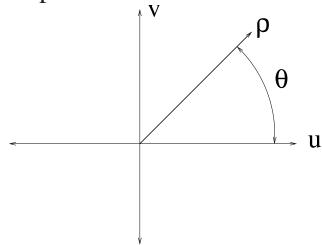
$$P_{\theta}(\rho) = CTFT \{p_{\theta}(r)\}$$

$$F(u, v) = CSFT \{f(x, y)\}$$

Then

$$P_{\theta}(\rho) = F(\rho \cos(\theta), \rho \sin(\theta))$$

• $P_{\theta}(\rho)$ is F(u, v) in polar coordinates!



Proof of the Fourier Slice Theorem

By definition

$$p_{\theta}(r) = \int_{-\infty}^{\infty} f\left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix}\right) dz$$

• The CTFT of $p_{\theta}(r)$ is then given by

$$P_{\theta}(\rho) = \int_{-\infty}^{\infty} p_{\theta}(r)e^{-j2\pi\rho r}dr$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix}\right) dz \right] e^{-j2\pi\rho r}dr$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\mathbf{A}_{\theta} \begin{bmatrix} r \\ z \end{bmatrix}\right) e^{-j2\pi\rho r}dzdr$$

• We next make the change of variables

$$\left[\begin{array}{c} r \\ z \end{array}\right] = \mathbf{A}_{-\theta} \left[\begin{array}{c} x \\ y \end{array}\right] .$$

Notice that the Jacobian is $|\mathbf{A}_{\theta}| = 1$, and that $r = x \cos(\theta) + y \sin(\theta)$. This results in

$$P_{\theta}(\rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi\rho[x\cos(\theta) + y\sin(\theta)]} dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi[x\rho\cos(\theta) + y\rho\sin(\theta)]} dxdy$$

$$= F(\rho\cos(\theta), \rho\sin(\theta))$$

Alternative Proof of the Fourier Slice Theorem

• First let $\theta = 0$, then

$$p_0(r) = \int_{-\infty}^{\infty} f(r, y) \, dy$$

Then

$$P_{0}(\rho) = \int_{-\infty}^{\infty} p_{0}(r)e^{-2\pi jr\rho} dr$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(r,y) dy \right] e^{-2\pi jr\rho} dr$$

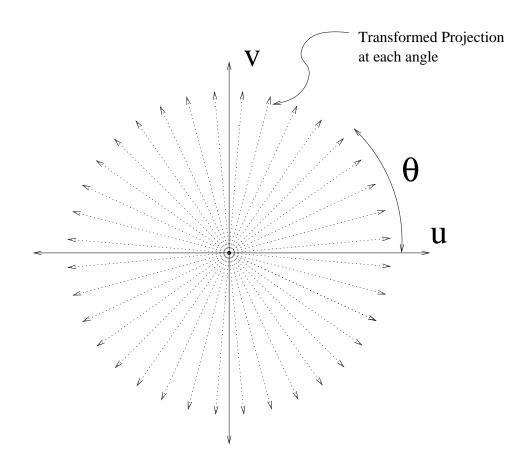
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r,y)e^{-2\pi j(r\rho+y0)} dr dy$$

$$= F(\rho,0)$$

• By rotation property of CSFT, it must hold for any θ .

Inverse Radon Transform

- Physical systems measure $p_{\theta}(r)$.
- From these, we compute $P_{\theta}(\rho) = CTFT\{p_{\theta}(r)\}.$



• Next we take an inverse CSFT to form f(x, y).

Problem: This requires polar to rectagular conversion.

Solution: Convolution backprojection

Convolution Back Projection (CBP) Algorithm

• In order to compute the inverse CSFT of F(u, v) in polar coordinates, we must use the Jacobian of the polar coordinate transformation.

$$du dv = |\rho| d\theta d\rho$$

• This results in the expression

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)e^{2\pi j(xu+yv)} dudv$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\pi} P_{\theta}(\rho)e^{2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))} |\rho|d\theta d\rho$$

$$= \int_{0}^{\pi} \left[\int_{-\infty}^{\infty} |\rho|P_{\theta}(\rho)e^{2\pi j\rho(x\cos(\theta)+y\sin(\theta))} d\rho \right] d\theta$$

$$g_{\theta}(x\cos(\theta)+y\sin(\theta))$$

 \bullet Then g(t) is given by

$$g_{\theta}(t) = \int_{-\infty}^{\infty} |\rho| P_{\theta}(\rho) e^{2\pi j \rho t} d\rho$$
$$= CTFT^{-1} \{ |\rho| P_{\theta}(\rho) \}$$
$$= h(t) * p_{\theta}(r)$$

where
$$h(t)=CTFT^{-1}\{|\rho|\}$$
, and
$$f(x,y)=\int_0^\pi g_\theta\left(x\cos(\theta)+y\sin(\theta)\right)d\theta$$

Summary of CBP Algorithm

- 1. Measure projections $p_{\theta}(r)$.
- 2. Filter the projections $g_{\theta}(r) = h(r) * p_{\theta}(r)$.
- 3. Back project filtered projections

$$f(x,y) = \int_0^{\pi} g_{\theta} (x \cos(\theta) + y \sin(\theta)) d\theta$$

A Closer Look at Projection Filter

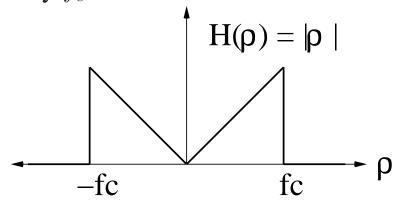
1. At each angle, projections are filtered.

$$g_{\theta}(r) = h(r) * p_{\theta}(r)$$

2. The frequency response of the filter is given by

$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to $|\rho| \leq f_c$ for some cut-off frequency f_c .



So

$$H(\rho) = f_c \left[\operatorname{rect} \left(f/(2f_c) \right) - \Lambda \left(f/f_c \right) \right]$$

$$h(r) = f_c^2 \left[2\operatorname{sinc}(t2f_c) - \operatorname{sinc}^2(tf_c) \right]$$

A Closer Look at Back Projection

• Back Projection function is

$$f(x,y) = \int_0^{\pi} b_{\theta}(x,y) d\theta$$

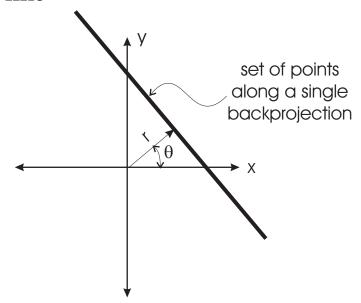
where

$$b_{\theta}(x, y) = g_{\theta}(x \cos(\theta) + y \sin(\theta))$$

ullet Consider the set of points (x, y) such that

$$r = x\cos(\theta) + y\sin(\theta)$$

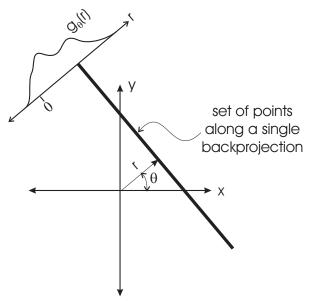
This set looks like



• Along this line $b_{\theta}(x,y) = g_{\theta}(r)$.

Back Projection Continued

• For each angle θ back projection is constant along lines of angle θ and takes on value $g_{\theta}(r)$.



• Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to π .

$$f(x,y) = \int_0^{\pi} b_{\theta}(x,y) d\theta$$
$$\approx \frac{\pi}{M} \sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x,y)$$

ullet Back projection "smears" values of g(r) back over image, and then adds smeared images for each angle.