

## Nonlinear Filtering

- Linear filters
  - Tend to blur edges and other image detail.
  - Perform poorly with non-Gaussian noise.
  - Result from Gaussian image and noise assumptions.
  - Images are not Gaussian.
- Nonlinear filter
  - Can preserve edges
  - Very effective at removing impulsive noise
  - Result from non-Gaussian image and noise assumptions.
  - Can be difficult to design.

## Linear Filters

- Definition: A system  $y = T[x]$  is said to be linear if for all  $\alpha, \beta \in \mathbb{R}$

$$\alpha y_1 + \beta y_2 = T[\alpha x_1 + \beta x_2]$$

where  $y_1 = T[x_1]$  and  $y_2 = T[x_2]$ .

- Any filter of the form

$$y_s = \sum_r h_{s,r} x_r$$

## Homogeneous Filters

- Definition: A filter  $y = T[x]$  is said to be homogeneous if for all  $\alpha \in \mathbb{R}$

$$\alpha y = T[\alpha x]$$

- This is much weaker than linearity.
- Homogeneity is a natural condition for scale invariant systems.

## Median Filter

- Let  $W$  be a window with an odd number of points.
- Then the median filter is given by

$$y_s = \text{median} \{x_{s+r} : r \in W\}$$

- Is the median filter:
  - Linear?
  - Homogeneous?
- Consider the 1-D median filter with a 3-point window.

x(m)	0	0	1	1,000	1	1	2	2
y(m)	?	0	1	1	1	1	2	?

## Median Filter: Optimization Viewpoint

- Consider the median filter

$$y_s = \text{median} \{x_{s+r} : r \in W\}$$

and consider the following functional.

$$F_s(\theta) \triangleq \sum_{r \in W} |\theta - x_{s+r}|$$

- Then  $y_s$  solves the following optimization equation.

$$y_s = \arg \min_{\theta} F_s(\theta)$$

- Differentiating, we have

$$\begin{aligned} \frac{d}{d\theta} F(\theta) &= \frac{d}{d\theta} \sum_{r \in W} |\theta - x_{s+r}| \\ &= \sum_{r \in W} \text{sign}(\theta - x_{s+r}) \\ &\triangleq f(\theta) \end{aligned}$$

This expression only holds for  $\theta \neq x_{s+r}$  for all  $r \in W$ .

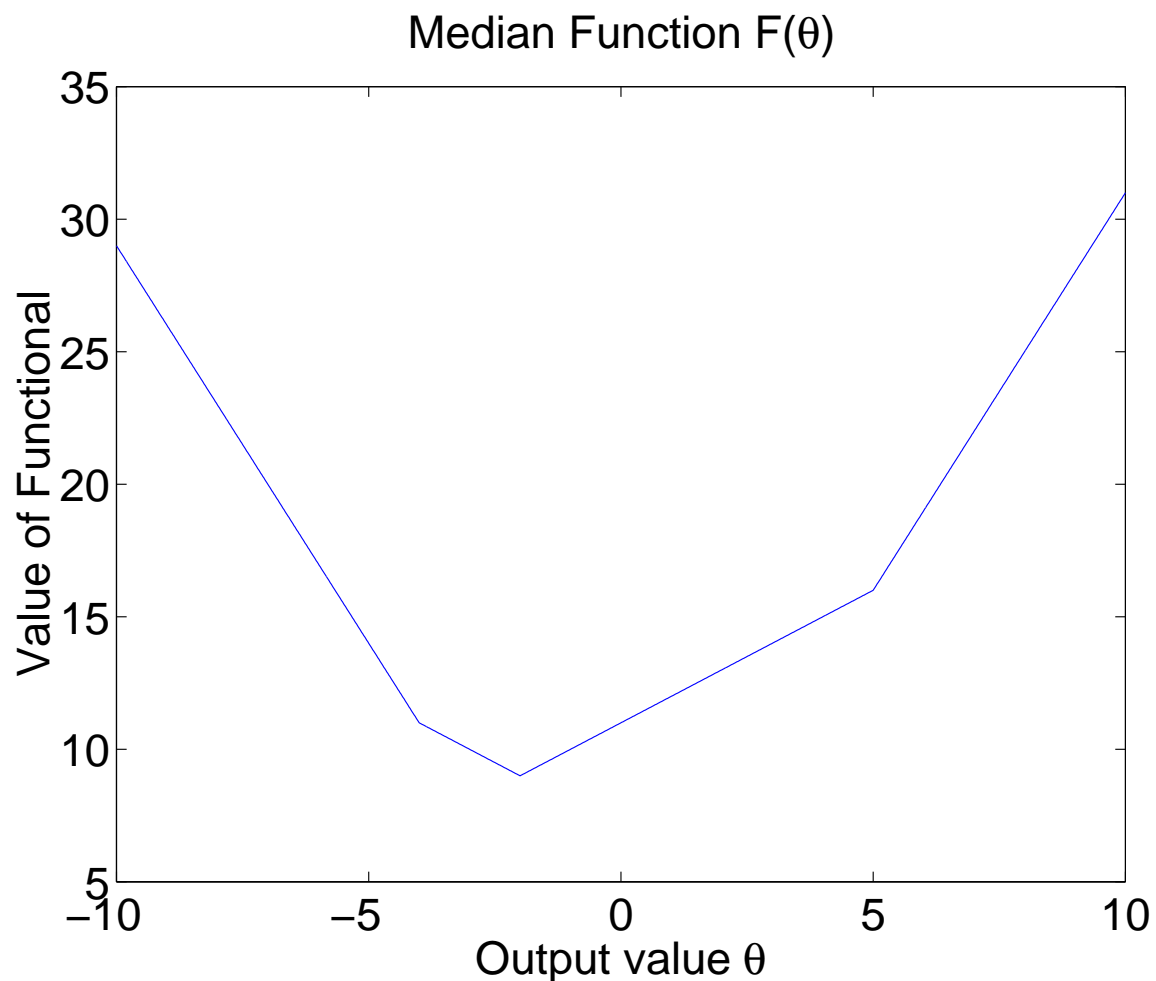
- So the solution falls at  $\theta = x_{s*}$  such that

$$0 = \sum_{\substack{r \in W \\ r \neq (s*-s)}} \text{sign}(\theta - x_{s+r})$$

## Example: Median Filter Function

- Consider a 1-D median filter
  - Three point window of  $W = \{-1, 0, 1\}$
  - Inputs  $[x(n-1), x(n), x(n+1)] = [-2, -4, 5]$ .

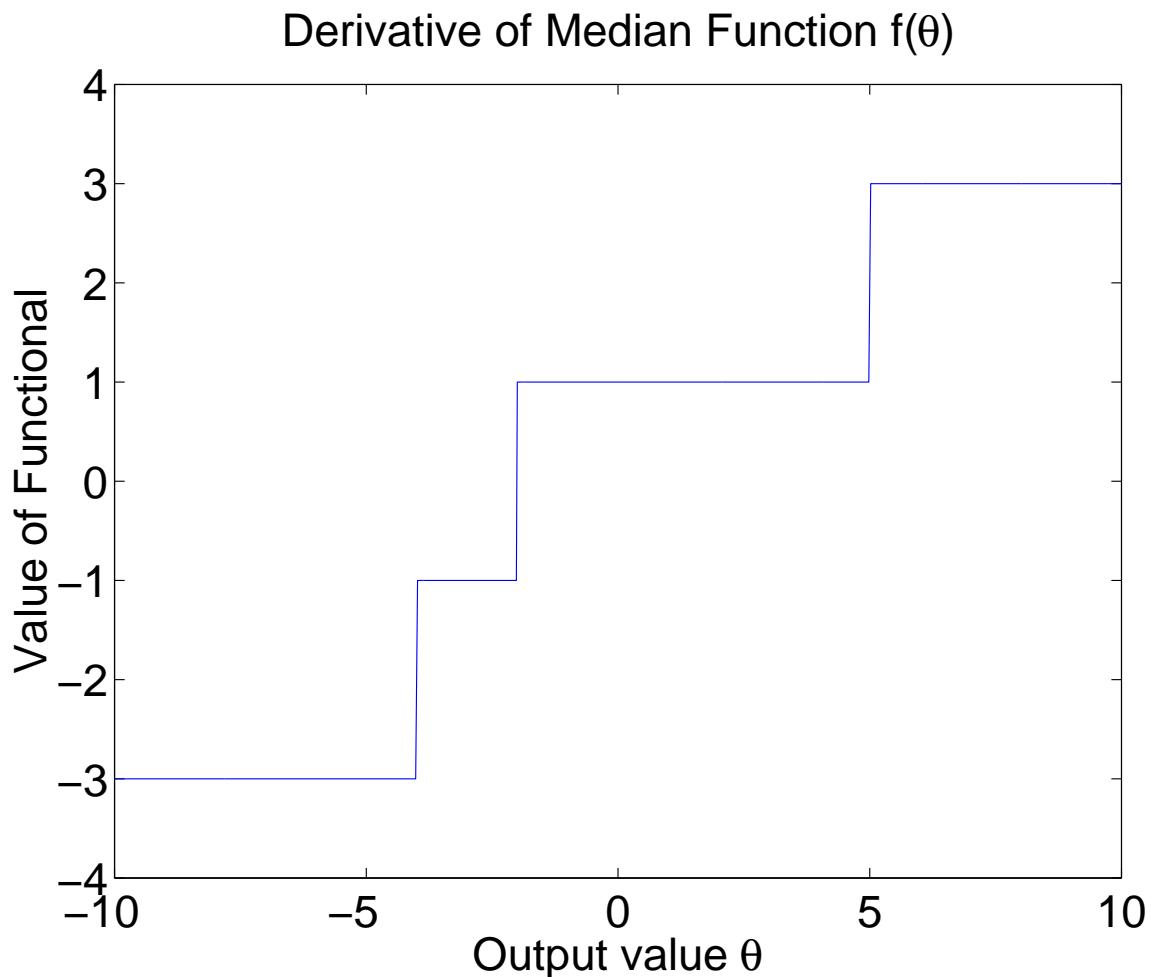
$$F(\theta) = \sum_{k=-1}^1 |\theta - x_{n+k}|$$



## Example: Derivative of Median Filter Function

- Consider a 1-D median filter
  - Three point window of  $W = \{-1, 0, 1\}$
  - Inputs  $[x(n-1), x(n), x(n+1)] = [-2, -4, 5]$ .

$$f(\theta) = \sum_{k=-1}^1 \text{sign}(\theta - x_{n+k})$$

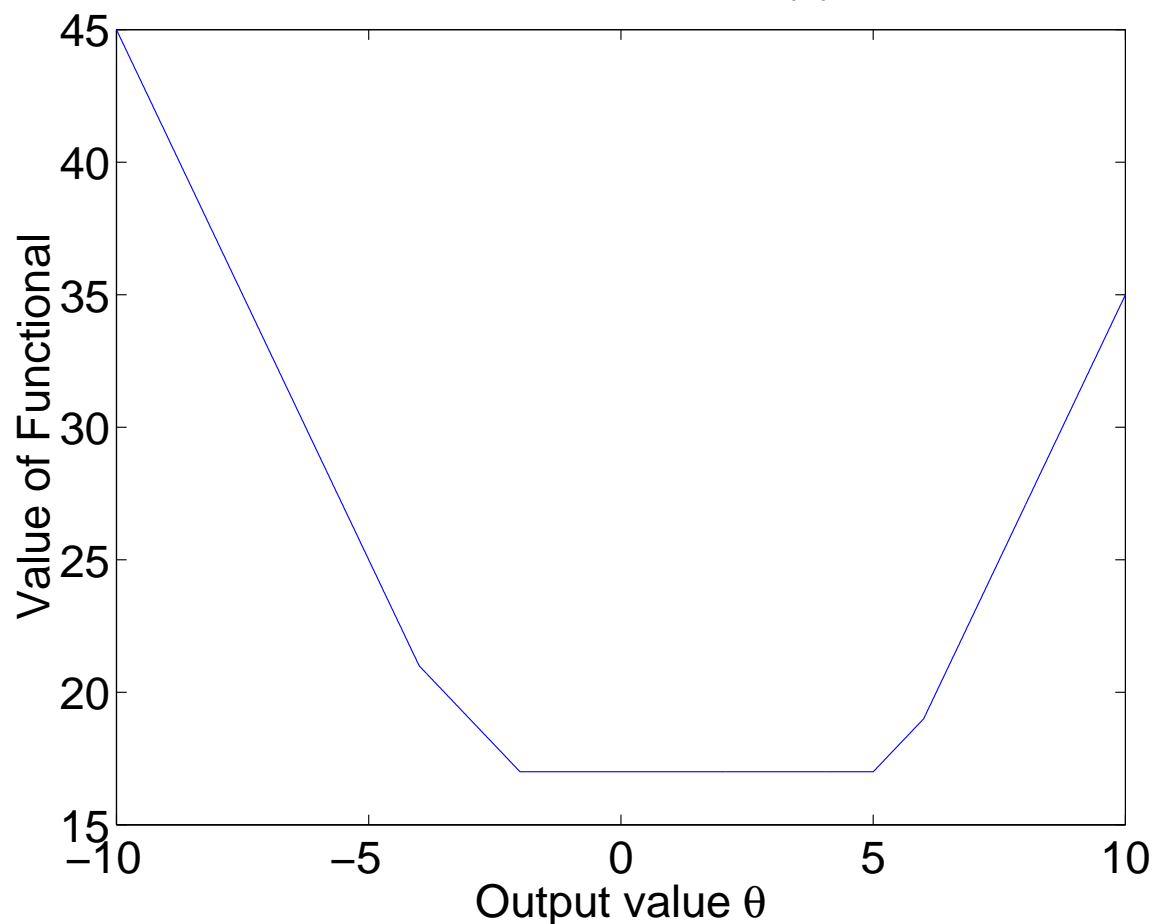


## Problem with an Even Number of Points

- Consider a 1-D median filter
  - Four point window of  $W = \{-1, 0, 1, 2\}$
  - Inputs  $[x(n-1), x(n), x(n+1), x(n+2)] = [-2, -4, 5, 6]$ .
- Solution is not unique.

$$F(\theta) = \sum_{k=-1}^2 |\theta - x_{n+k}|$$

Median Function  $F(\theta)$





## Weighted Median Filter

- Defined the functional

$$F(\theta) \triangleq \sum_{r \in W} a_r |\theta - x_{s+r}|$$

where  $a_r$  are weights assigned to each point in the window  $W$ .

- Weighted median is computed by

$$y_s = \arg \min_{\theta} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

- Differentiating, we have

$$\begin{aligned} \frac{d}{d\theta} F(\theta) &= \frac{d}{d\theta} \sum_{r \in W} a_r |\theta - x_{s+r}| \\ &= \sum_{r \in W} a_r \text{sign}(\theta - x_{s+r}) \\ &\triangleq f(\theta) \end{aligned}$$

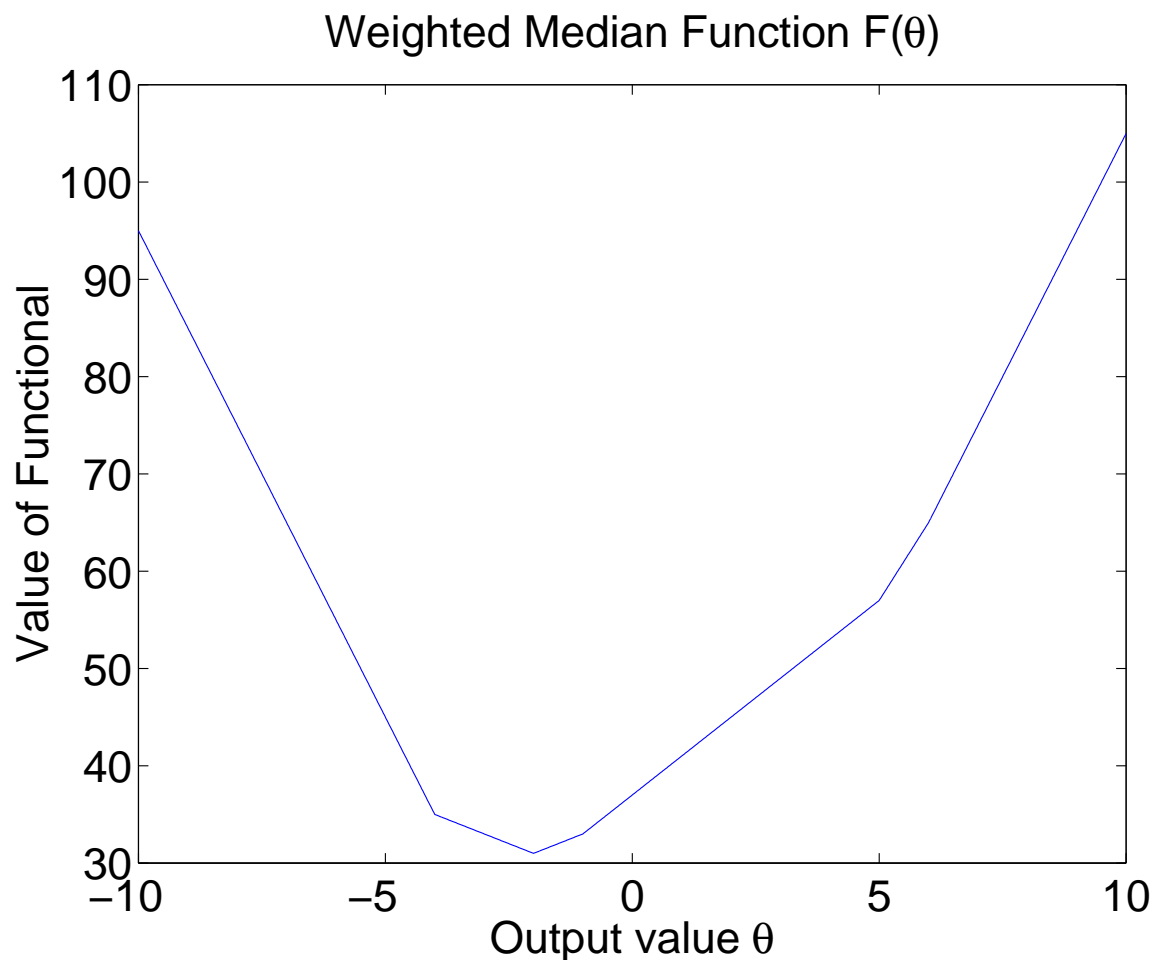
This expression only holds for  $\theta \neq x_r$  for all  $r \in W$ .

- Need to find  $s^*$  such that  $f(\theta)$  is “nearly” zero.

## Example: Weighted Median Filter Function

- Consider a 1-D median filter
  - Five point window of  $W = \{-2, -1, 0, 1, 2\}$
  - Inputs  $[x(n-2), \dots, x(n+2)] = [6, -2, -4, 5, -1]$ .
  - Weights  $[a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1]$ .

$$F(\theta) = \sum_{k=-1}^1 a(k) |\theta - x_{n+k}|$$

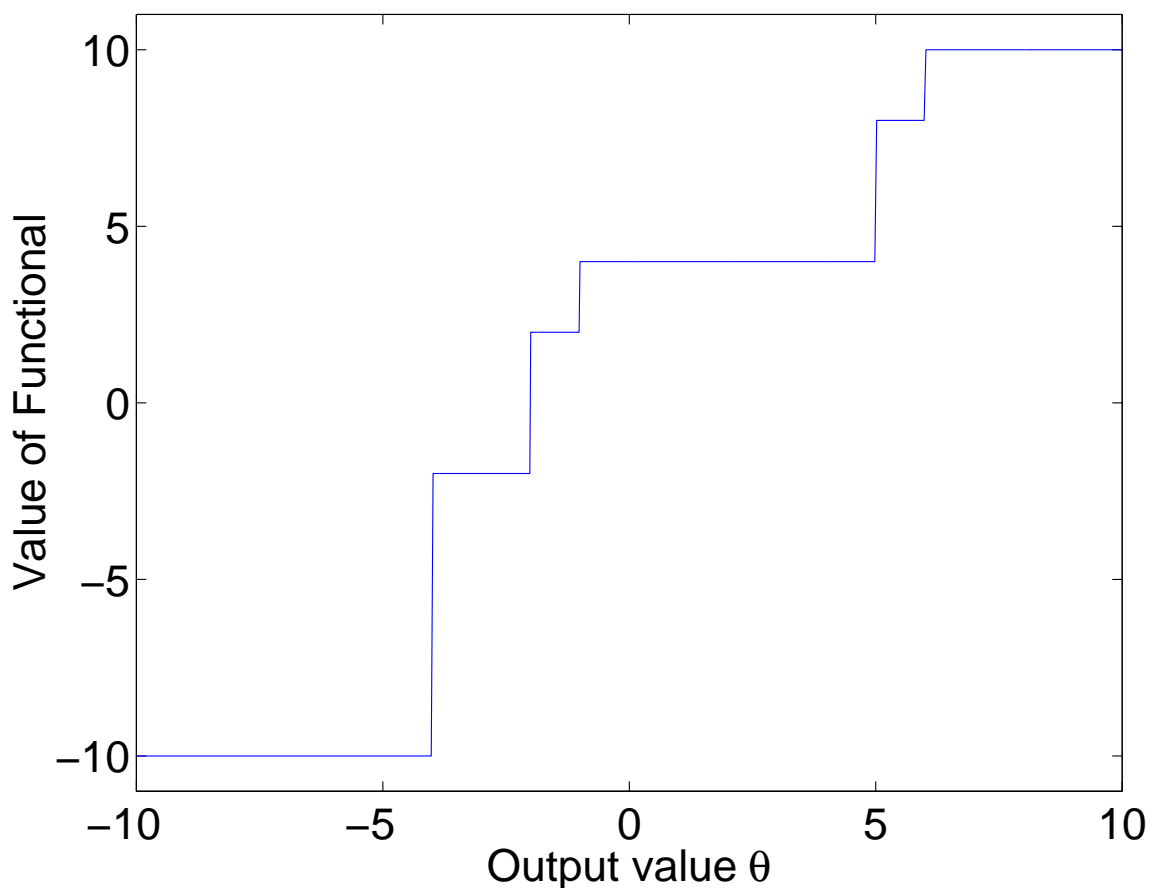


## Example: Derivative of Median Filter Function

- Consider a 1-D median filter
  - Five point window of  $W = \{-2, -1, 0, 1, 2\}$
  - Inputs  $[x(n-2), \dots, x(n+2)] = [6, -2, -4, 5, -1]$ .
  - Weights  $[a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1]$ .

$$f(\theta) = \sum_{k=-1}^1 a(k) \text{sign}(\theta - x_{n+k})$$

Derivative of Weighted Median Function  $f(\theta)$



## Computation of Weighted Median

1. Sort points in window.

- Let  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(p)}$  be the sorted values.
- These values are known as order statistics.
- Let  $a_{(1)}, a_{(2)}, \dots, a_{(p)}$  be the **corresponding** weights.

2. Find  $i^*$  such that the following equations hold

$$\begin{aligned}
 a_{i^*} + \sum_{i=1}^{i^*-1} a_{(i)} &\geq \sum_{i=i^*+1}^p a_{(i)} \\
 \sum_{i=1}^{i^*-1} a_{(i)} &\leq \sum_{i=i^*+1}^p a_{(i)} + a_{i^*}
 \end{aligned}$$

3. Then the value  $x_{(i^*)}$  is the weighted median value.

## Comments on Weighted Median Filter

- Weights may be adjusted to yield the “best” filter.
- Largest and smallest values are ignored.
- Same as median filter for  $a_r = 1$ .