EE 637 Final

May 1, Spring 2013

Name: ____

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}} [X(f)]$$

$$\mathrm{rep}_T [x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(20pt)

Consider the following 2-D LSI systems. The first system has input $x(m,n)$ and output $y(m,n)$, and the second system has input $y(m,n)$ and output $z(m,n)$.

$$
\begin{aligned}
y(m,n) &= \sum_{j=-N}^{N} a_j \, x(m, n-j) && \text{S1} \\
z(m,n) &= \sum_{i=-N}^{N} b_i \, y(m-i, n) && \text{S2}
\end{aligned}
$$

a) Calculate the 2-D impulse response, $h_1(m,n)$, of the first system.

b) Calculate the 2-D impulse response, $h_2(m,n)$, of the second system.

c) Calculate the 2-D impulse response, $h(m,n)$, of the complete system.

d) How many multiplies does it take per output point to implement each of the two individual systems? How, many multiplies does it take per output point to implement the complete system with a single convolution.

e) Explain the advantages and disadvantages of implementing the two systems in sequence versus a single complete system.

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Problem 2.(20pt)

Consider a gray level image, $g(m,n)$, which takes values on the interval $[0,1]$, and let $b(m,n)$ be a corresponding random binary image that is constrained to take on the value 0 or 1 at each point. Also, define the display error as $d(m,n) = b(m,n) - g(m,n)$.

a) Specify the two probabilities $P\{b(m,n) = 0\} = p_0(m,n)$ and $P\{b(m,n) = 1\} = p_1(m,n)$ that in ensure that $E[b(m,n)] = g(m,n)$.

b) Using the values of $p_0(m,n)$ and $p_1(m,n)$, calculate the variance of $b(m,n)$.

For parts c) through e), let $T(m,n)$ be a set of i.i.d. random thresholds used for ordered dither of the image; so that

$$b(m,n) = \begin{cases} 1 & \text{if } g(m,n) \geq T(m,n) \\ 0 & \text{if } g(m,n) < T(m,n) \end{cases} .$$

c) Determine a distribution of $T(m,n)$ that ensures that the $E[b(m,n)] = g(m,n)$.

d) Calculate $E[d(m,n)]$ and $R(m,n,k,l) = E[d(m,n)d(m+k,n+l)]$.

e) Will $b(m,n)$ be a good quality halftone for the gray level g? Justify your answer.

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Problem 3.(20pt)

Let X_n be a discrete-time random process with i.i.d. samples, and distribution given by $P\{X_n = k\} = p_k$ where

$$(p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7) = \left(\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right) \ ,$$

and let

$$Y_n = \begin{cases} 0 & \text{if } X_n \leq 4 \\ 1 & \text{if } X_n > 4 \end{cases} \quad .$$

a) Calculate the distribution and entropy of Y_n.

b) Calculate the entropy $H(X_n)$ in bits.

c) Draw the Huffman tree and determine the binary Huffman code for each possible symbol.

d) Calculate the expected code length per symbol.

e) Are there better codes for X_n? If so, what are they? If not, why not?

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Problem 4.(20pt)

Consider a columnated beam of X-ray photons passing through an absorbing material, (i.e., a thin beam of X-rays that form a column as they pass through a material). The columnated X-rays then pass in a straight line through an object of length T with density $u(x)$ where x is the depth into the object. The number of photons in the beam at depth $x \in [0, T]$ is denoted by the random variable Y_x with Poisson density given by

$$P\{Y_x = k\} = \frac{e^{-\lambda_x}\lambda_x^k}{k!} \ ,$$

where all distances are measured in units of cm and all absorption constants are measured in units of cm^{-1}.

Furthermore, assume that the random variable Y_T is measured at the exit of the material, and from this measurement, our objective will be to estimate the projection

$$p = \int_0^T \mu(x) dx \ .$$

a) Write a differential equation which describes the behavior of λ_x as a function of x. (Hint: It should have the form $\frac{d\lambda_x}{dx} = \textit{something}$, and *something* should be a function of $\mu(x)$.)

b) Solve the differential equation of part a) to find an expression for λ_x as a function x in terms of $\mu(x)$.

c) Determine an estimate, $\hat{p} = f(Y_T)$, so that $p \approx \hat{p}$.

d) Is the estimate unbiased, i.e., does the following equation hold?

$$E[\hat{p}] \stackrel{?}{=} p$$

e) Determine an approximate expression for the variance of $\hat{p}$.

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Problem 5.(20pt)

Consider a 2D Gaussian random vector, X, with distribution $N(0, R)$, and covariance $R = E\Lambda E^t$ where $\Lambda = diag\{16, 1\}$, and

$$E = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

a) Determine a transformation, $X = TY$, where $Y = [Y_1, Y_2]^t$, where Y_1 and Y_2 are independent Gaussian random variables with mean 0 and variance 1.

b) For $\theta = \pi/3$, sketch a contour plot of the 2D distribution of X with X_1 as the horizontal axis and X_2 as the vertical axis.

c) Write the density function for the vector Y, and the vector X.

d) Write an expression for the rate-distoration of the random vector X.

e) Sketch the theoretical lower-bound on the distortion versus the rate for an encoder of X.

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