## Wiener-Khintchine Theorem

For a well behaved stationary random process the power spectrum is equal to the Fourier transform of the autocorrelation function.

$$S_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}$$

Sloppy proof:

$$\begin{aligned}
S_x(e^{j\omega}) &= \lim_{N\to\infty} \frac{1}{2N+1} E[|X_N(e^{j\omega})|^2] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} E\left[\left(\sum_{n=-N}^{N} x(n) e^{-j\omega n}\right)\left(\sum_{k=-N}^{N} x(k) e^{-j\omega k}\right)^*\right] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} E\left[\sum_{n=-N}^{N}\sum_{k=-N}^{N} x(n)x(k) e^{-j\omega(n-k)}\right] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\sum_{k=-N}^{N} E[x(n)x(k)] e^{-j\omega(n-k)} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\sum_{k=-N}^{N} R_x(n-k) e^{-j\omega(n-k)} \\
&= \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N}\sum_{k=-M}^{M} R_x(n-k) e^{-j\omega(n-k)} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lim_{M\to\infty} \sum_{k=-M}^{M} R_x(n-k) e^{-j\omega(n-k)} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}\right) \\
&= \left(\sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}\right) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 \\
&= \sum_{k=-\infty}^{\infty} R_x(k) e^{-j\omega k}
\end{aligned}$$