Random Variables

- Let X be a random variable on $\mathbb{R}$, then
  - X is usually denoted by an upper case letter.
  - The cummulative distribution function is given by

$$P\{X \leq x\} = F_X(x)$$

  - If the probability density function exists, it is given by

$$p_X(x) = \frac{dF_X(x)}{dx}$$

so that

$$\begin{aligned} P\{x_1 < X \leq x_2\} &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} p_X(\tau) d\tau \end{aligned}$$

  - The expectation of X is given by

$$E[X] = \int_{-\infty}^{\infty} \tau p_X(\tau) d\tau$$

or more precisely by the Riemann-Stieltjes integral

$$E[X] = \int_{-\infty}^{\infty} \tau dF_X(\tau)$$

if it exists.

Deterministic versus Random

- Let X and Z be random variables, and let $f(\cdot)$ be a function from $\mathbb{R}$ to $\mathbb{R}$
  - Is $Y = f(X)$ a random variable?
  - Is $\mu = E[X]$ a random variable?
  - Is $\hat{X} = E[X|Z]$ a random variable?

Properties of Expectation

- Expectation is linear

$$E[X+Y] = E[X] + E[Y]$$

- What is $E[E[X|Y]]$ equal to?

$$E[E[X|Y]] = E[X]$$

- What is $E[X|X,Y]$ equal to?

$$E[X|X,Y] = X$$

- When X, Y, and Z are (jointly) Gaussian

$$E[X|Y,Z] = aY + bZ + c$$

for some scalar values a, b, and c.

2-D Discrete Space Random Processes

- Notation
  - X_s is a pixel at position $s = (s_1, s_2) \in \mathcal{Z}^2$
  - S denotes the set of 2-D Lattice points where $S \subset \mathcal{Z}^2$
- Definitions
  - Mean $\mu_s = E[X_s]$
  - Autocorrelation $R_{sr} = E[X_s X_r]$
  - Autocovariance $C_{sr} = E[(X_s - \mu_s)(X_r - \mu_r)]$
  - A process is said to be **second order** if $E[X_s]$ and $E[X_s X_r]$ exist for all $s \in S$ and $r \in S$.
  - A second order random process is said to be **wide sense stationary** if for all $s \in \mathcal{Z}^2$

$$\begin{aligned} \mu_s &= \mu_{(0,0)} \\ C_{r,r+s} &= C_{(0,0),s} \end{aligned}$$

2-D Power Spectral Density

Let X_s be a zero mean wide sense stationary random process.

Define

$$\hat{X}_N(e^{j\mu}, e^{j\nu}) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} X_{(m,n)} e^{-j(m\mu + n\nu)}$$

- Then the power spectrum (i.e. energy spectrum per unit sample) is

$$\frac{1}{(2N+1)^2} \left| \hat{X}_N(e^{j\mu}, e^{j\nu}) \right|^2$$

The following limit does not converge!!

$$\lim_{N \to \infty} \frac{1}{(2N+1)^2} \left| \hat{X}_N(e^{j\mu}, e^{j\nu}) \right|^2$$

Intuition - The spectral estimate remains noisy as the window size increases.

Definition of Power Spectral Density

- Definition of **Power Spectral Density**

$$S_x(e^{j\mu}, e^{j\nu}) \triangleq \lim_{N\to\infty} \frac{1}{(2N+1)^2} E\left[\left|\hat{X}_N(e^{j\mu}, e^{j\nu})\right|^2\right]$$

Expectation removes the noise.

Weiner-Khintchine Theorem

- For a wide sense stationary random process, the power spectral density equals the Fourier transform of the autocorrelation

$$S_x(e^{j\mu}, e^{j\nu}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m,n) e^{-j(m\mu+n\nu)}$$

where

$$R(m,n) = E[X_{(0,0)} X_{(m,n)}]$$

Estimating Power Spectral Density

- For simplicity, consider 1-D case

$$S_x(e^{j\omega}) = \lim_{N\to\infty} \frac{1}{N} E\left[\left|\hat{X}_N(e^{j\omega})\right|^2\right]$$

How do we compute the required expectation?

  – Answer: The law of large numbers (averaging)

$$E[Z] = \lim_{N\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} Z_k$$

where Z_k are independent and identically distributed (i.i.d.).

Computing Power Spectrum from Block Average

- Let X_n be signal with $0 \leq n < NK$.
- Break X_n into K parts, each of length N.

$$Y_n^{(k)} = X_{kN+n}$$

where $0 \leq k < K$ and $0 \leq n < N$.

- Compute DTFT of $Y_n^{(k)}$

$$\hat{Y}^{(k)}(e^{j\omega}) = \sum_{n=0}^{N-1} Y_n^{(k)} e^{j\omega n}$$

- Average power spectrum estimates for each block

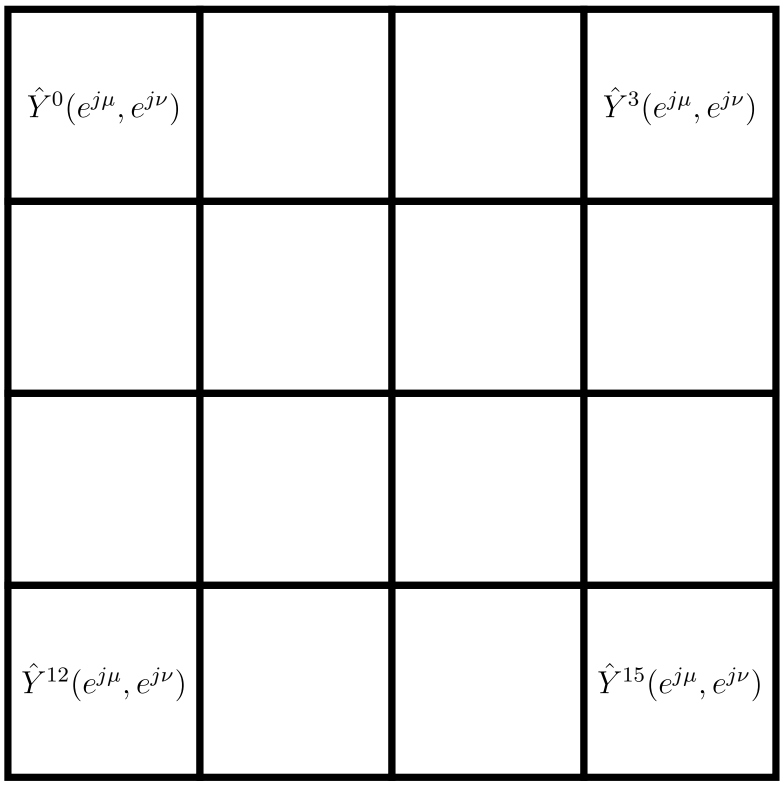
$$\begin{aligned} S_x(e^{j\omega}) &= \frac{1}{N} E\left[\left|\hat{Y}^{(k)}(e^{j\omega})\right|^2\right] \\ &\cong \frac{1}{N}\left[\frac{1}{K}\sum_{k=0}^{K-1}\left|\hat{Y}^{(k)}(e^{j\omega})\right|^2\right] \end{aligned}$$

- So we have that

$$S_x(e^{j\omega}) \cong \frac{1}{NK}\sum_{k=0}^{K-1}\left|\hat{Y}^{(k)}(e^{j\omega})\right|^2$$

Block Averaging in 2-D

- Break image into K regions each with N pixels



- For each block, compute $\hat{Y}^{(k)}(e^{j\mu}, e^{j\nu})$
- Average blocks to form power spectrum estimate

$$S_x(e^{j\mu}, e^{j\nu}) \cong \frac{1}{NK} \sum_{k=0}^{K-1} \left| \hat{Y}^{(k)}(e^{j\mu}, e^{j\nu}) \right|^2$$