

## Image Restoration

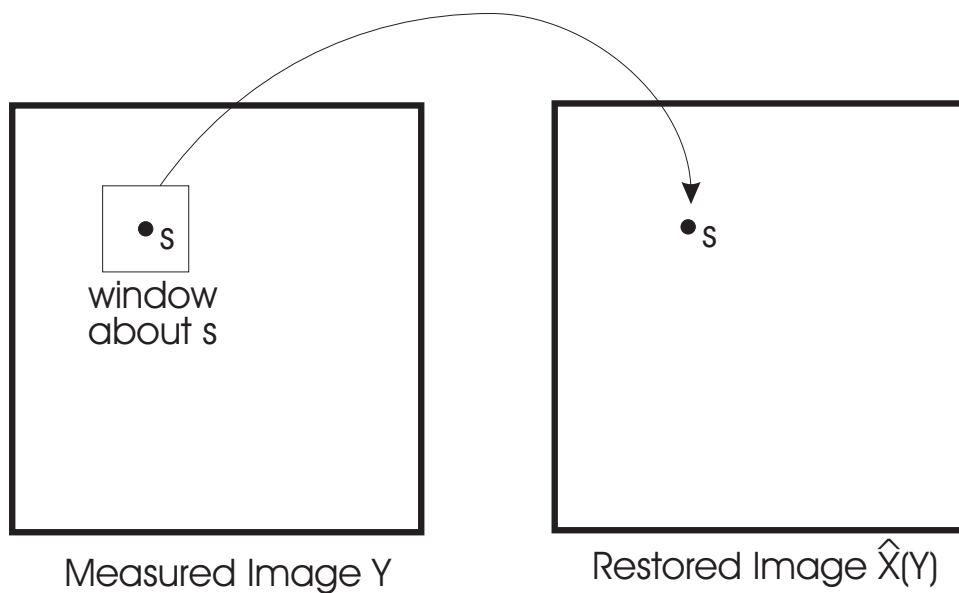
- Problem:
  - You want to know some image  $X$ .
  - But you only have a corrupted version  $Y$ .
  - How do you determine  $X$  from  $Y$ ?
- Corruption may result from:
  - Additive noise
  - Nonadditive noise
  - Linear distortion
  - Nonlinear distortion

## Optimum Linear FIR Filter

- Find an “optimum” linear filter to compute  $X$  from  $Y$ .
- Filter uses input window of  $Y$  to estimate each output pixel  $X_s$ .
- Filter can be designed to be minimize mean squared error (MMSE).
- The estimate of  $X_s$  is denoted by  $\hat{X}_s$ .
- $W(s)$  denotes the window about  $s$ .
- The estimate,  $\hat{X}_s$ , is a function of  $Y_{W(s)}$ .

## Application of Optimum Filter

$$\hat{X}_s = f(Y_{W(s)})$$



- The function  $f(Y_{W(s)})$  is designed to produce a MMSE estimate of  $X$ .
- If  $f(Y_{W(s)})$  is:
  - Linear  $\Rightarrow$  linear space invariant filter.
  - Nonlinear  $\Rightarrow$  nonlinear space invariant filter.
- This filter can reduce the effects of all types of corruption.

## Optimality Properties of Linear Filter

- If both images are jointly Gaussian:
  - Then MMSE filter is linear.

$$\begin{aligned}\hat{X}_s &= E[X_s | Y_{W(s)}] \\ &= \mathbf{A} Y_{W(s)} + b\end{aligned}$$

- If images are not jointly Gaussian:
  - Then MMSE filter is generally not linear.

$$\begin{aligned}\hat{X}_s &= E[X_s | Y_{W(s)}] \\ &= f(Y_{W(s)})\end{aligned}$$

- However, the MMSE linear filter can still be very effective!

## Formulation of MMSE Linear Filter: Definitions

- $W(s)$  - window about the pixel  $s$ .
- $p$  - number of pixels in  $W(s)$
- $z_s$  - row vector containing pixels of  $Y_{W(s)}$ .
- $\theta$  - column containing filter parameters
- Detailed definitions:

- Definition of  $W(s)$

$$W(s) = [s, s + r_1, \dots, s + r_{p-1}]$$

where  $r_1, \dots, r_{p-1}$  index neighbors.

- Definition of  $z_s$

$$z_s = [y_s, y_{s+r_1}, \dots, y_{s+r_{p-1}}]$$

- Definition of  $\theta$

$$\theta = [\theta_0, \dots, \theta_{p-1}]$$

## Formulation of MMSE Linear Filter: Objectives

- Linear filter is given by

$$\hat{x}_s = z_s \theta$$

- Mean squared error is given by

$$\begin{aligned} MSE &= E[|x_s - \hat{x}_s|^2] \\ &= E[|x_s - z_s \theta|^2] \end{aligned}$$

- The MMSE filter parameters  $\theta^*$  are given by

$$\theta^* = \arg \min_{\theta} E[|x_s - z_s \theta|^2] .$$

- How do we solve this problem?

## More Matrix Notation

- Define the subset  $S_0$  of image pixels.
  1.  $S_0 \subset S$
  2.  $S_0$  contains  $N_0 < N$  pixels
  3.  $S_0$  usually does not contain pixels on the boundary of the image.
  4.  $S_0 = [s_1, \dots, s_{N_0}]$
- Define the  $N_0 \times p$  matrix  $Z$

$$Z = \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix}.$$

- Define the  $N_0 \times 1$  column vectors  $X$  and  $\hat{X}$

$$X = \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{bmatrix} \hat{x}_{s_1} \\ \hat{x}_{s_2} \\ \vdots \\ \hat{x}_{s_{N_0}} \end{bmatrix}.$$

- Then

$$X \approx \hat{X} = Z\theta$$

## Least Squares Linear Filter

- We expect that

$$\begin{aligned}MSE &= E[|x_s - z_s\theta|^2] \\&\approx \frac{1}{N_0} \sum_{s \in S_0} |x_s - z_s\theta|^2 \\&= \frac{1}{N_0} \|X - Z\theta\|^2\end{aligned}$$

- So we may solve the equation

$$\theta^* = \arg \min_{\theta} \|X - Z\theta\|^2$$

- The solution  $\theta^*$  is the least squares estimate, of  $\theta$ , and the estimate

$$\hat{X} = Z\theta^*$$

is known as the least squares filter.



## Computing Least Squares Linear Filter

$$\theta^* = \arg \min_{\theta} \frac{1}{N_0} \|X - Z\theta\|^2$$

• So

$$\begin{aligned} \theta^* &= \arg \min_{\theta} \left( \frac{1}{N_0} \|X - Z\theta\|^2 \right) \\ &= \arg \min_{\theta} \left( \frac{1}{N_0} (X - Z\theta)^t (X - Z\theta) \right) \\ &= \arg \min_{\theta} \left( \frac{1}{N_0} (X^t X - 2\theta^t Z^t X + \theta^t Z^t Z \theta) \right) \\ &= \arg \min_{\theta} \left( \frac{X^t X}{N_0} - 2\theta^t \frac{Z^t X}{N_0} + \theta^t \frac{Z^t Z}{N_0} \theta \right) \\ &= \arg \min_{\theta} \left( \theta^t \frac{Z^t Z}{N_0} \theta - 2\theta^t \frac{Z^t X}{N_0} \right) \end{aligned}$$

## Covariance Estimates

- Define the  $p \times p$  matrix

$$\begin{aligned}
 \hat{R}_{zz} &\triangleq \frac{Z^t Z}{N_0} \\
 &= \frac{1}{N_0} \begin{bmatrix} z_{s_1}^t & z_{s_2}^t & \dots & z_{s_{N_0}}^t \end{bmatrix} \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} \\
 &= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t z_{s_i}
 \end{aligned}$$

- Define the  $p \times 1$  vector

$$\begin{aligned}
 \hat{r}_{zx} &\triangleq \frac{Z^t X}{N_0} \\
 &= \frac{1}{N_0} \begin{bmatrix} z_{s_1}^t & z_{s_2}^t & \dots & z_{s_{N_0}}^t \end{bmatrix} \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \\
 &= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t x_{s_i}
 \end{aligned}$$

- So

$$\theta^* = \arg \min_{\theta} \left( \theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

## Interpretation of $\hat{R}_{zz}$ and $\hat{r}_{zx}$

- $\hat{R}_{zz}$  is an estimate of the covariance of  $z_s$ .

$$\begin{aligned} E[\hat{R}_{zz}] &= E\left[\frac{1}{N_0} \sum_{s=1}^N z_s^t z_s\right] \\ &= E[z_s^t z_s] \\ &= R_{zz} \end{aligned}$$

- $\hat{r}_{zx}$  is an estimate of the cross correlation between  $z_s$  and  $x_s$ .

$$\begin{aligned} E[\hat{r}_{zx}] &= E\left[\frac{1}{N_0} \sum_{s=1}^N z_s^t x_s\right] \\ &= E[z_s^t x_s] \\ &= r_{zx} \end{aligned}$$

## Solution to Least Squares Linear Filter

- We need

$$\theta^* = \arg \min_{\theta} \frac{1}{N_0} \|X - Z\theta\|^2$$

We have shown this is equivalent to

$$\theta^* = \arg \min_{\theta} \left( \theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

- Taking the gradient of the cost functional

$$\begin{aligned} 0 &= \nabla_{\theta} \left( \theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right) \Big|_{\theta=\theta^*} \\ &= \left( 2\hat{R}_{zz} \theta - 2\hat{r}_{zx} \right) \Big|_{\theta=\theta^*} \end{aligned}$$

Solving for  $\theta^*$  yeilds

$$\theta^* = \left( \hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

## Summary of Solution to Least Squares Linear Filter

- First compute

$$\hat{R}_{zz} = \frac{1}{N_0} \sum_{s=1}^N z_s^t z_s$$

$$\hat{r}_{zx} = \frac{1}{N_0} \sum_{s=1}^N z_s^t x_s$$

- Then compute

$$\theta^* = \left( \hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

- The vector  $\theta^*$  then contains the values of the filter coefficients.

## Training

- $\theta^*$  is usually estimated from “training” data.
- Training data
  - Generally consists of image pairs  $(X, Y)$  where  $Y$  is the measured data and  $X$  is the undistorted image.
  - Should be typical of what you might expect.
  - Can often be difficult to obtain.
- Testing data
  - Also consists of image pairs  $(X, Y)$ .
  - Is used to evaluate the effectiveness of the filters.
  - Should never be taken from the training data set.
- Training versus Testing
  - Performance on training data is always better than performance on testing data.
  - As the amount of training data increases, the performance on training and testing data both approach the best achievable performance.

## Comments

- Wiener filter is the MMSE **linear** filter.
- Wiener filter may be optimal, but it isn't always good.
  - Linear filters blur edges
  - Linear filters work poorly with non-Gaussian noise.
- Nonlinear filters can be designed using the same methodologies.

## Is MMSE a Good Quality Criteria for Images?

- In general, NO! ... But sometimes it is OK.
- For achromatic images, it is best to choose  $X$  and  $Y$  in  $L^*$  or gamma corrected coordinates.
- Let  $H$  be a filter that implements the CSF for the human visual system.
  - Then a better metric of error is

$$\begin{aligned}
 HVSE &= ||H(X - \hat{X})||^2 \\
 &= (X - \hat{X})^t H^t H (X - \hat{X}) \\
 &= ||X - \hat{X}||_B^2
 \end{aligned}$$

where  $B = H^t H$ .

- $||X - \hat{X}||_B^2$  is a quadratic norm.
- What is the minimum HVSE estimate  $\hat{X}$ ?



## Answer

- The answer is  $\hat{X} = E[X|Y]$ .
  - This is the same as for mean squared error!
  - The conditional expectation minimizes any quadratic norm of the error.
  - This is also true for non-Gaussian images.
- Let  $\hat{X} = AY_{W(s)} + b$  be the MMSE **linear** filter.
  - This filter is also the minimum HVSE **linear** filter.
  - This is also true for non-Gaussian images.

## Proof

- Define  $V \triangleq HX$  and  $B = H^t H$

$$\begin{aligned}
 & \min_{\hat{X}} E \left[ ||X - \hat{X}||_B^2 \right] \\
 &= \min_{\hat{X}} E \left[ ||H(X - \hat{X})||^2 \right] \\
 &= \min_{\hat{V}} E \left[ ||V - \hat{V}||^2 \right] \\
 &= E \left[ ||V - E[V|Y]||^2 \right] \\
 &= E \left[ ||HX - E[HX|Y]||^2 \right] \\
 &= E \left[ ||H(X - E[X|Y])||^2 \right] \\
 &= E \left[ ||X - E[X|Y]||_B^2 \right]
 \end{aligned}$$

- So,  $\hat{X} = E[X|Y]$  minimizes the error measure.

$$HVSE = ||X - \hat{X}||_B^2 .$$