EE 637 Final

May 2, Spring 2012

Name: Key

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \stackrel{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \stackrel{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \stackrel{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \stackrel{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \stackrel{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \stackrel{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \stackrel{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \stackrel{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \stackrel{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy \\ f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \stackrel{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\text{rep}_T [x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\text{comb}_T [x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\text{comb}_T [x(t)] \stackrel{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}} [X(f)]$$

$$\text{rep}_T [x(t)] \stackrel{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}} [X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1 First Part.(8pt)

Let $X \sim N(0, \sigma_x^2)$ and $W \sim N(0, 1)$ be independent Gaussian random variables, and let

$$Y = X + W \; .$$

Furthermore, we know that the MMSE estimate of X given Y has a form given by

$$\hat{X} = E[X|Y] = \alpha Y \; ,$$

and we know that the MMSE estimate must have the property that

$$E\left[(X - \hat{X})Y\right] = 0 \; .$$

a) Calculate the $E[XY]$ and $E[Y^2]$.

b) Calculate the value of α.

Problem ~~1~~ Second Part.(~~8~~pt) 12pt

Let $X \sim N(0, R)$ and $W \sim N(0, I)$ be independent Gaussian random vectors with $R = E\Lambda E^t$ where $EE^t = I$ and Λ is diagonal, and let

$$Y = X + W \; .$$

Use the results of Problem 1) First Part to solve the following problems.

c) Calculate the mean and covariance of the three Gaussian random vectors $\tilde{X} = E^t X$, $\tilde{W} = E^t W$, and $\tilde{Y} = E^t Y$.

d) Calculate an expression for the $Z = E\left[\tilde{X}|\tilde{Y}\right]$.

e) Use the expression of part d) to calculate an expression for the $E[X|Y]$.

a) $E[XY] = E[X(X+W)] = \sigma_x^2 + 0$

$E[Y^2] = E[(X+W)(X+W)] = \sigma_x^2 + 2E[XW] + \sigma_w^2$

$= \sigma_x^2 + \sigma_w^2 = \sigma_x^2 + 1$

b) $E[(X-\hat{X})Y] = 0 = E[(X - \alpha Y)Y]$

$= E[XY] - \alpha E[Y^2]$

$$\alpha = \frac{E[XY]}{E[Y^2]}$$

$$\alpha = \frac{\sigma_x^2}{\sigma_x^2 + 1}$$

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c) $E[\tilde{X}\tilde{X}^*] = E[E^* X X^* E]$

$$= E^* E[XX^*] E = E^* R E$$

$$= E^* E \Lambda E^* E = \Lambda$$

$E[\tilde{W}\tilde{W}^*] = I$

$E[\tilde{Y}\tilde{Y}^*] = \Lambda + I$

d) $E[(\tilde{X} - Z)\tilde{Y}^*] = 0$

$E[(\tilde{X} - \Theta\tilde{Y})\tilde{Y}^*] = 0$

$p \times p$ matrix

$E[\tilde{X}\tilde{Y}^*] - \Theta E[\tilde{Y}\tilde{Y}^*] = 0$

$\Lambda - \Theta(\Lambda + I) = 0$

$\Theta = (\Lambda + I)^{-1}\Lambda$

$Z = (\Lambda + I)^{-1}\Lambda\tilde{Y}$

e) $\hat{X} = E[X \mid Y]$

$= EZ = E(\Lambda + I)^{-1}\Lambda E^* Y$

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Problem 2.(20pt)

Consider the 2D system given by

$$g(x,y) = f(x,y) \circ h(x,y)$$

where $\circ$ represents 2D convolution and

$$h(x,y) = \frac{1}{\sqrt{x^2+y^2}} \ .$$

A common problem in image processing is called "deconvolution". The objective of deconvolution is to recover the signal $f(x,y)$ from knowledge of $g(x,y)$ and $h(x,y)$.

a) Use the Fourier transform to show that

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi rt} dr \ .$$

b) Use the result of a) to show that

$$\delta(u\cos\theta + v\sin\theta) = \int_{-\infty}^{\infty} e^{j2\pi r(u\cos\theta + v\sin\theta)} dr \ .$$

c) Show that the Fourier transform of $h(x,y)$ has the form

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\pi/2}^{\pi/2} \frac{1}{r} e^{j2\pi(ur\cos\theta + vr\sin\theta)} r\,d\theta\,dr = \int_{-\pi/2}^{\pi/2} \delta(u\cos\theta + v\sin\theta)d\theta$$

d) Use this result to show that

$$H(u,v) = \frac{1}{\sqrt{u^2+v^2}} \ .$$

e) Which task is more difficult: Deconvolution with $h(x,y) = \frac{1}{\sqrt{x^2+y^2}}$; or Deconvolution with $h(x,y) = \mathrm{sinc}(x,y)$? Justify your answer.

a) $\int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi t} dt = 1 \Rightarrow \int_{-\infty}^{\infty} 1\, e^{j2\pi\omega t} d\omega = \delta(t)$

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi rt} dr$$

b) $t \rightarrow u\cos\theta + v\sin\theta$

$$\delta(u\cos\theta + v\sin\theta) = \int_{-\infty}^{\infty} e^{j2\pi r(u\cos\theta + v\sin\theta)} dr$$

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c) $H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$

since $h(x,y) = h(-x,-y)$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, e^{+j2\pi(ux+vy)}\, dx\, dy$

in polar coordinates

$H(u,v) = \int_{-\infty}^{\infty}\int_{-\pi/2}^{\pi/2} \cancel{h}\, e^{j2\pi(ur\cos\theta + vr\sin\theta)}\, r\, d\theta\, dr$

$= \int_{0}^{\pi}\int_{-\infty}^{\infty} e^{j2\pi r(u\cos\theta + v\sin\theta)}\, dr\, d\theta$

$= \int_{0}^{\pi} \delta(u\cos\theta + v\sin\theta)\, d\theta$

d) let $(u,v) = (\rho\cos\phi, \rho\sin\phi)$

$\rho = \sqrt{u^2+v^2} \quad \phi = \arctan(u,v)$

$= \int_{0}^{\pi} \delta(\rho\cos(\theta-\phi))\, d\theta$

$= \frac{1}{\rho} = \frac{1}{\sqrt{u^2+v^2}}$

e) ~~E~~asier to deconvolve with $\frac{1}{\sqrt{x^2+y^2}}$ because the spectrum falls off slower.

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Problem 3.(20pt)

Consider the 1D interpolation system given by

$$s(n) = \sum_{m=-\infty}^{\infty} x(m)\delta(n - mL)$$

and

$$y(n) = \sum_{m=-\infty}^{\infty} s(n-m)h(m) \ .$$

a) **Derive** an expression for $S(e^{j\omega})$ in terms of $X(e^{j\omega})$.

b) Write an expression for the ideal filter $H(e^{j\omega})$.

c) Derive an expression for the ideal filter impulse response $h(n)$.

d) Is the ideal $h(n)$ practical to use on streaming audio sources? Why or why not?

e) Suggest an alternative impulse response $h(n)$ to use, and explain its advantages.

a) $$S(e^{j\omega}) = \sum_{n=-\infty}^{\infty} s(n)\, e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n L}$$

$$= X(e^{j\omega L})$$

b) $$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{L} \\ 0 & \text{for } \frac{\pi}{L} \le |\omega| < \pi \end{cases}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} L\, \mathrm{rect}\left(\frac{\omega - k2\pi}{2\pi/L}\right)$$

c) $$h(n) = \mathrm{sinc}(n/L)$$

d) No, because it requires infinite number of multiplies per output sample.

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e) $h(n) = w(n)\ \mathrm{sinc}(n/L)$

some window such as Hamming window with appropriate shift to center it at $n=0$.

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Problem 4.(20pt)

Consider a Gaussian random variable X with mean zero and variance 1.

a) Write an expression for the rate and distortion functions $R(\delta)$ and $D(\delta)$.

b) Sketch the distortion-rate function for X.

c) Explain the meaning of the distortion-rate function.

d) Let $\{X_n\}_{n=1}^N$ be a set of independent Gaussian random variables each with zero mean and variance σ_n^2 where $\sigma_{n-1}^2 \geq \sigma_n^2$. Write an expression for the rate and distortion.

e) How can one calculate the distortion-rate function for a Gaussian vector X with mean zero and covariance $R = E\Lambda E^t$ where $EE^t = I$ and Λ is diagonal?

a) $R(\delta) = \max\left\{\frac{1}{2}\log_2 \frac{1}{\delta}, 0\right\}$

$D(\delta) = \min\{1, \delta\}$

b)

c) It is the lower bound on achievable rates at a given distortion. Given $D' > D(\delta)$ and $R' > R(\delta)$, one can find a coder that achieves rate R' and distortion R'.

d) $R(\delta) = \sum_{n=1}^{N} \max\left\{\frac{1}{2}\log_2\left(\frac{\sigma_n^2}{\delta}\right), 0\right\}$

$D(\delta) = \sum_{n=1}^{N} \min\{\sigma_n^2, \delta\}$

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e) $$R(\delta) = \sum_{n=1}^{N} \max\left\{ \frac{1}{2} \log\left(\frac{\Lambda_{nn}}{\delta} \right), 0 \right\}$$

$$D(\delta) = \sum_{n=1}^{N} \min\left\{ \Lambda_{nn}, \delta \right\}$$

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Problem 5.(20pt)

Let $X(m,n)$ be an achromatic image that is collected by a linear CCD with subtraction of dark current so that $X(m,n) = 0$ corresponds to zero light photons. Furthermore, assume there is a gamma-corrected display devise with inputs $R(m,n)$, $G(m,n)$, and $B(m,n)$ taking integer values ranging from 0 to 255.

a) Specify the (approximate) transforms of the image $X(m,n)$ that are necessary in order to display the image properly assuming that the display takes standard RGB input.

b) If you were to display the image $X(m,n)$ directly so that $R(m,n) = G(m,n) = B(m,n) = X(m,n)$, then what distortion would you observe in the displayed image? Again, assume the display takes standard RGB input.

c) If the display were modified so that its $\gamma = 1$, and you were to display the image $X(m,n)$ directly so that $R(m,n) = G(m,n) = B(m,n) = X(m,n)$, then what distortion would you observe in the displayed image?

d) Imagine that $X(m,n)$ is collected from a half-toned print, and you would like to remove the halftone noise. Is it better to filter $X(m,n)$ directly, or to first gamma correct $X(m,n)$ and then filter it? Justify your answer.

e) What is your favorite color?

a) $$R(m,n) = G(m,n) = B(m,n) = 255\left(\frac{X(m,n)}{X_{max}}\right)^{\frac{1}{2.2}}$$

X_{max} = maximum sensor output

b) It would be too dark.

c) You would see quantization artifacts in the dark regions.

d) It is better to filter $X(m,n)$. Filtering the gamma corrected version will produce a shift in brightness.