# Types of Coding

- Source Coding - Code data to more efficiently represent the information
  - Reduces "size" of data
  - Analog - Encode analog source data into a binary format
  - Digital - Reduce the "size" of digital source data
- Channel Coding - Code data for transmition over a noisy communication channel
  - Increases "size" of data
  - Digital - add redundancy to identify and correct errors
  - Analog - represent digital values by analog signals
- Complete "Information Theory" was developed by Claude Shannon

# Digital Image Coding

- Images from a 6 MPixel digital cammera are 18 MBytes each
- Input and output images are digital
- Output image must be smaller (i.e. $\approx$ 500 kBytes)
- This is a digital source coding problem

# Two Types of Source (Image) Coding

- Lossless coding (entropy coding)
  - Data can be decoded to form exactly the same bits
  - Used in "zip"
  - Can only achieve moderate compression (e.g. 2:1 - 3:1) for natural images
  - Can be important in certain applications such as medical imaging
- Lossly source coding
  - Decompressed image is visually similar, but has been changed
  - Used in "JPEG" and "MPEG"
  - Can achieve much greater compression (e.g. 20:1 - 40:1) for natural images
  - Uses entropy coding

# Entropy

- Let $X$ be a random variables taking values in the set $\{0, \cdots, M-1\}$ such that

$$p_i = P\{X = i\}$$

- Then we define the entropy of $X$ as

$$\begin{aligned} H(X) &= -\sum_{i=0}^{M-1} p_i \log_2 p_i \\ &= -E\left[\log_2 p_X\right] \end{aligned}$$

  $H(X)$ has units of bits

# Conditional Entropy and Mutual Information

- Let $(X, Y)$ be a random variables taking values in the set $\{0, \cdots, M-1\}^2$ such that

$$p(i,j) \;=\; P\{X = i, Y = j\}$$

$$p(i|j) \;=\; \frac{p(i,j)}{\Sigma_{k=0}^{M-1} p(k,j)}$$

- Then we define the conditional entropy of $X$ given $Y$ as

$$H(X|Y) \;=\; -\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} p(i,j)\log_2 p(i|j)$$

$$= \; -E\left[\log_2 p(X|Y)\right]$$

- The mutual information between $X$ and $Y$ is given by

$$I(X;Y) = H(X) - H(X|Y)$$

  The mutual information is the reduction in uncertainty of $X$ given $Y$.

# Entropy (Lossless) Coding of a Sequence

- Let $X_n$ be an i.i.d. sequence of random variables taking values in the set $\{0, \cdots, M-1\}$ such that

$$P\{X_n = m\} = p_m$$

  - $X_n$ for each $n$ is known as a symbol

- How do we represent $X_n$ with a minimum number of bits per symbol?

# A Code

- **Definition:** A code is a mapping from the discrete set of symbols $\{0, \cdots, M-1\}$ to finite binary sequences
  - For each symbol, $m$ their is a corresponding finite binary sequence $\sigma_m$
  - $|\sigma_m|$ is the length of the binary sequence
- Expected number of bits per symbol (bit rate)

$$\begin{aligned}\bar{n} &= E\left[|\sigma_{X_n}|\right] \\ &= \sum_{m=0}^{M-1} |\sigma_m|\, p_m\end{aligned}$$

- Example for $M = 4$

| m | $\rightarrow$ | $\sigma_m$ | $\lvert\sigma_m\rvert$ |
|---|---|---|---|
| 0 | $\rightarrow$ | $0\,1$ | 2 |
| 1 | $\rightarrow$ | $1\,0$ | 2 |
| 2 | $\rightarrow$ | $0$ | 1 |
| 3 | $\rightarrow$ | $1\,0\,0\,1\,0\,0$ | 6 |

  - Encoded bit stream

$$(0, 2, 1, 3, 2) \rightarrow (01|0|10|100100|0)$$

# Fixed versus Variable Length Codes

- Fixed Length Code - $|\sigma_m|$ is constant for all $m$
- Variable Length Code - $|\sigma_m|$ varies with $m$

- Problem
  - Variable length codes may not be uniquely decodable
  - Example: Using code from previous page

$$(6) \rightarrow (100100)$$

$$(1, 0, 2, 2) \rightarrow (10|01|0|0)$$

  - Different symbol sequences can yield the same code

- **Definition:** A code is *Uniquely Decodable* if there exists only a single unique decoding of each coded sequence.
- **Definition:** A *Prefix Code* is a specific type of uniquely decodable code in which no code is a prefix of another code.

# Lower Bound on Bit Rate

- **Theorem:** Let $C$ be a uniquely decodable code for the i.i.d. symbol sequence $X_n$. Then the mean code length is greater than $H(X_n)$.

$$\begin{aligned}\bar{n} &\stackrel{\triangle}{=} e\left[\left|\sigma_{X_n}\right|\right] \\ &= \sum_{m=0}^{M-1} \left|\sigma_m\right| p_m \\ &\geq H(X_n)\end{aligned}$$

- Quetion: Can we achieve this bound?

- Answer: Yes! Constructive proof using Huffman codes

# Huffman Codes

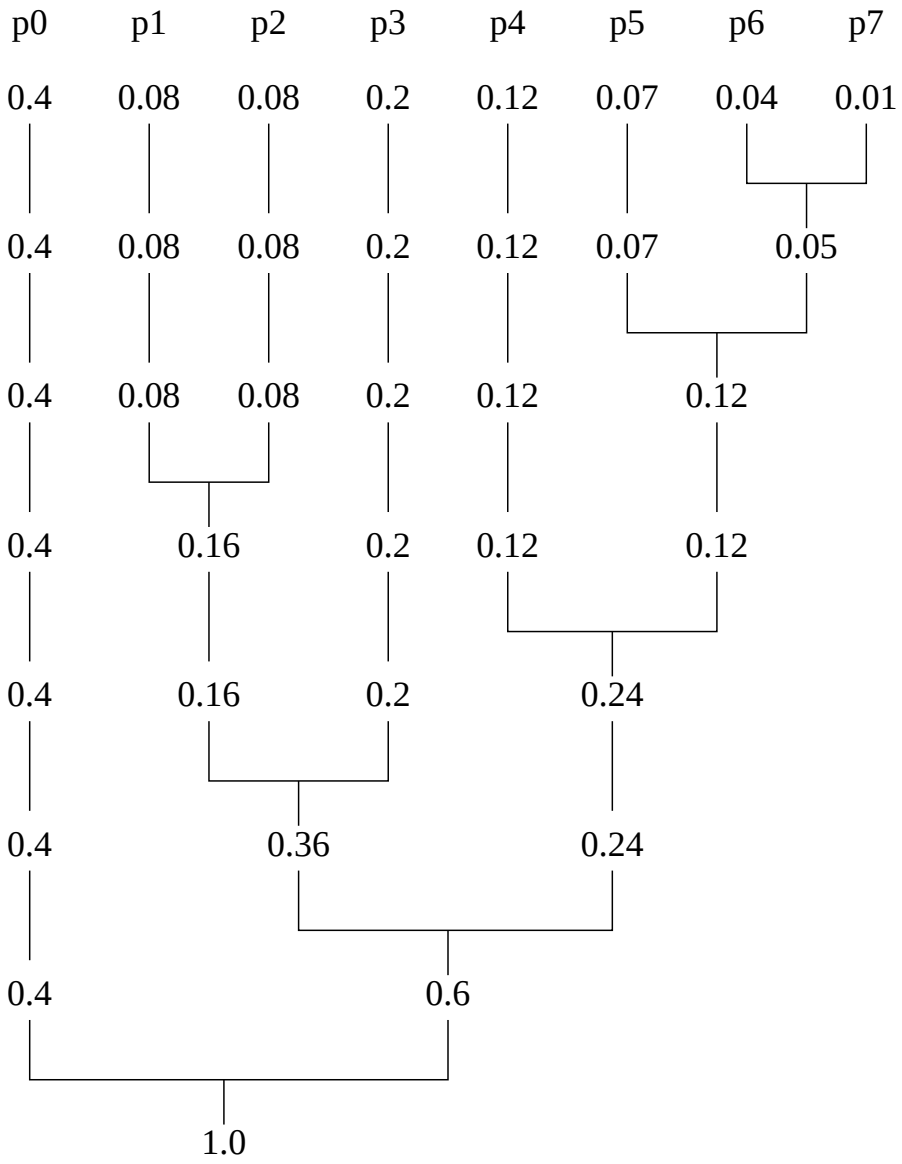
- Variable length prefix code $\Rightarrow$ Uniquely decodable
- Basic idea:
  - Low probability symbols $\Rightarrow$ Long codes
  - High probability symbols $\Rightarrow$ short codes
- Basic algorithm:
  - Low probability symbols $\Rightarrow$ Long codes
  - High probability symbols $\Rightarrow$ short codes

# Huffman Coding Algorithm

1. Initialize list of probabilities with the probability of each symbol
2. Search list of probabilities for two smallest probabilities, $p_{k*}$ and $p_{l*}$.
3. Add two smallest probabilities to form a new probability, $p_m = p_{k*} + p_{l*}$.
4. Remove $p_{k*}$ and $p_{l*}$ from the list.
5. Add $p_m$ to the list.
6. Go to step 2 until the list only contains 1 entry

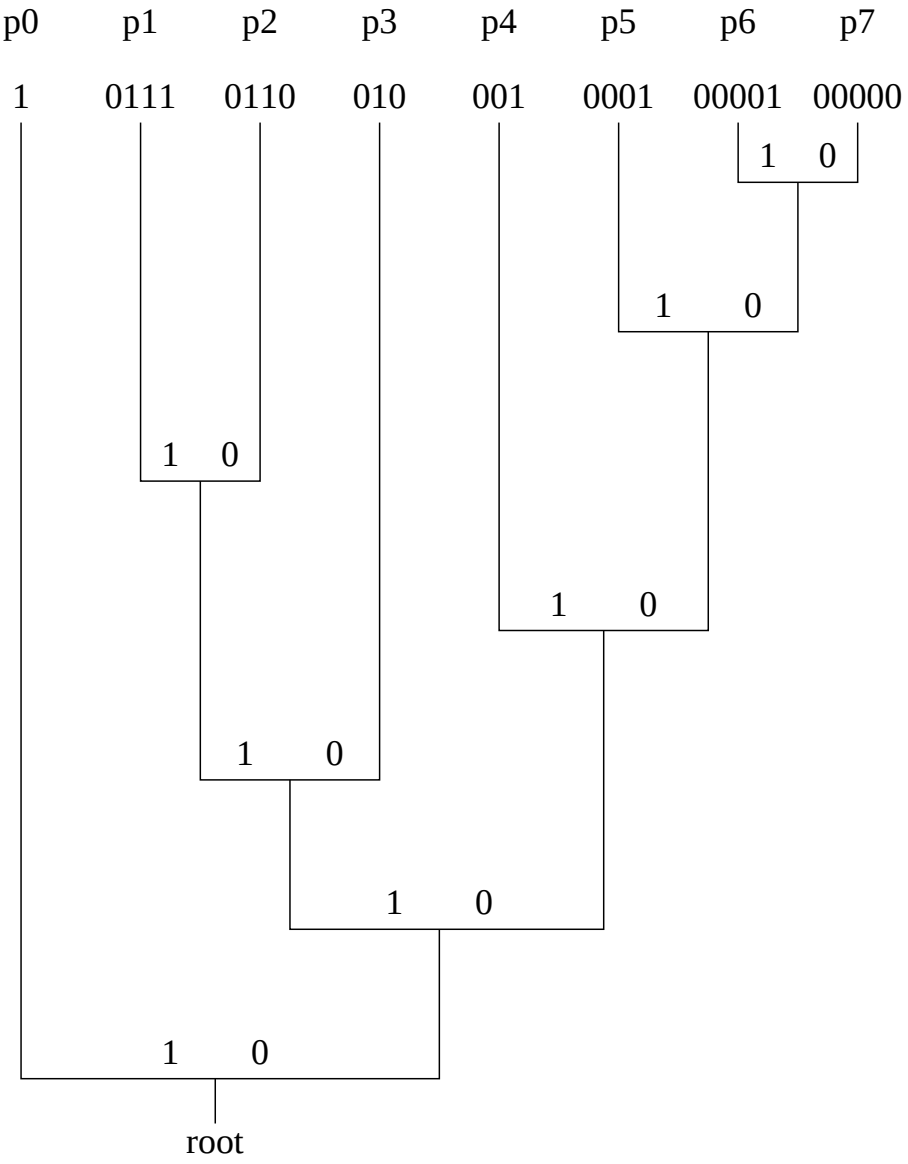
# Recursive Merging for Huffman Code

- Example for $M = 8$ code



# Resulting Huffman Code

- Binary codes given by path through tree



# Upper Bound on Bit Rate of Huffman Code

- **Theorem:** For a Huffman code, $\bar{n}$ has the property that

$$H(X_n) \leq \bar{n} < H(X_n) + 1$$

- A Huffman code is within 1 bit of optimal efficiency
- Can we do better?

# Coding in Blocks

- We can code blocks of symbols to achieve a bit rate that approaches the entropy of the source symbols.

$$\cdots, \underbrace{X_0, \cdots, X_{m-1}}_{Y_0}, \underbrace{X_m, \cdots, X_{2m-1}}_{Y_1}, \cdots$$

So we have that

$$Y_n = \left[X_{nm}, \cdots, X_{(n+1)m-1}\right]$$

where

$$Y_n \in \{0, \cdots, M^m - 1\}$$

# Bit Rate Bounds for Coding in Blocks

- It is easily shown that $H(Y_n) = mH(X_n)$ and the number of bits per symbol $X_n$ is given by $\bar{n}_x = \frac{\bar{n}_y}{m}$ where $\bar{n}_y$ is the number of bits per symbol for a Huffman code of $Y_n$.
- Then we have that

$$H(Y_n) \leq \bar{n}_y < H(Y_n) + 1$$

$$\frac{1}{m}H(Y_n) \leq \frac{\bar{n}_y}{m} < \frac{1}{m}H(Y_n) + \frac{1}{m}$$

$$H(X_n) \leq \frac{\bar{n}_y}{m} < H(X_n) + \frac{1}{m}$$

$$H(X_n) \leq \bar{n}_x < H(X_n) + \frac{1}{m}$$

- As the block size grows, we have

$$\lim_{m\to\infty} H(X_n) \leq \lim_{m\to\infty} \bar{n}_x \leq H(X_n) + \lim_{m\to\infty} \frac{1}{m}$$

$$H(X_n) \leq \lim_{m\to\infty} \bar{n}_x \leq H(X_n)$$

- So we see that for a Huffman code of blocks with length $m$

$$\lim_{m\to\infty} \bar{n}_x = H(X_n)$$

# Comments on Entropy Coding

- As the block size goes to infinity the bit rate approaches the entropy of the source

$$\lim_{m\to\infty} \bar{n}_x = H(X_n)$$

- A Huffman coder can achieve this performance, but it requires a large block size.
- As $m$ becomes large $M^m$ becomes very large $\Rightarrow$ large blocks are not practical.
- This assumes that $X_n$ are i.i.d., but a similar result holds for stationary and ergodic sources.
- Arithmetic coders can be used to achieve this bitrate in practical situations.

# Run Length Coding

- In some cases, long runs of symbols may occur. In this case, run length coding can be effective as a preprocessor to an entropy coder.
- Typical run length coder uses

  $$\cdots, (\text{value}, \#\text{ of repetitions}), (\text{value}, \#\text{ of repetitions+1}), \cdots$$

  where $2^b$ is the maximum number of repetitions
- Example: Let $X_n \in \{0, 1, 2\}$

  $$\cdots \,|\, \underbrace{0000000}_{07} \,|\, \underbrace{111}_{13} \,|\, \underbrace{222222}_{26} \,|\, \cdots$$

- If more than $2^b$ repetitions occur, then the repetition is broken into segments

  $$\cdots \,|\, \underbrace{00000000}_{08} \,|\, \underbrace{00}_{02} \,|\, \underbrace{111}_{13} \,|\, \cdots$$

- Many other variations are possible.

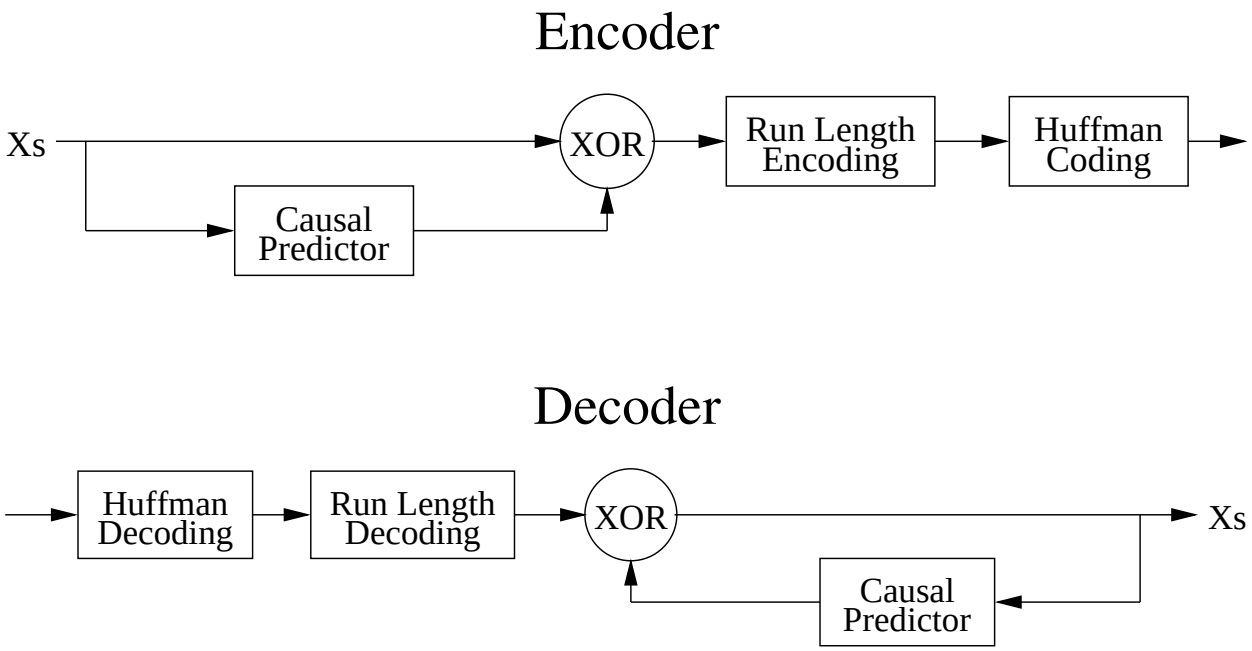
# Predictive Entropy Coder for Binary Images

- Uses in transmission of Fax images (CCITT G4 standard)
- Framework
  - Let $X_s$ be a binary image on a rectangular lattice $s = (s_1, s_2) \in S$
  - Let $W$ be a causal window in raster order
  - Determine a model for $p(x_s | x_{s+r}\, r \in W)$
- Algorithm
  1. For each pixel in raster order
     (a) Predict

$$\hat{X}_s = \begin{cases} 1 & \text{if } p(1|X_{s+r}\, r \in W) > p(0|X_{s+r}\, r \in W) \\ 0 & \text{otherwise} \end{cases}$$

     (b) If $X_s = \hat{X}_s$ send 0; otherwise send 1
  2. Run length code the result
  3. Entropy code the result

# Predictive Entropy Coder Flow Diagram



## How to Choose $p(x_s | x_{s+r}\, r \in W)$?

- Non-adaptive method
  - Select typical set of training images
  - Design predictor based on training images
- Adaptive method
  - Allow predictor to adapt to images being coded
  - Design decoder so it adapts in same manner as encoder

# Non-Adaptive Predictive Coder

- Method for estimating predictor
  1. Select typical set of training images
  2. For each pixel in each image, form $z_s = (x_{s+r_0}, \cdots, x_{s+r_{p-1}})$ where $\{r_0, \cdots, r_{p-1}\} \in W$.
  3. Index the values of $z_s$ from $j = 0$ to $j = 2^p - 1$
  4. For each pixel in each image, compute

$$h_s(i,j) = \delta(x_s = i)\delta(z_s = j)$$

     and the histogram

$$h(i,j) = \sum_{s \in S} h_s(i,j)$$

  5. Estimate $p(x_s | x_{s+r}\, r \in W) = p(x_s|z_s)$ as

$$\hat{p}(x_s = i | z_s = j) = \frac{h(i,j)}{\Sigma_{k=0}^{1} h(k,j)}$$

# Adaptive Predictive Coder

- Adapt predictor at each pixel
- Update value of $h(i,j)$ at each pixel using equations

$$h(i,j) \leftarrow h(i,j) + \delta(x_s = i)\delta(z_s = j)$$

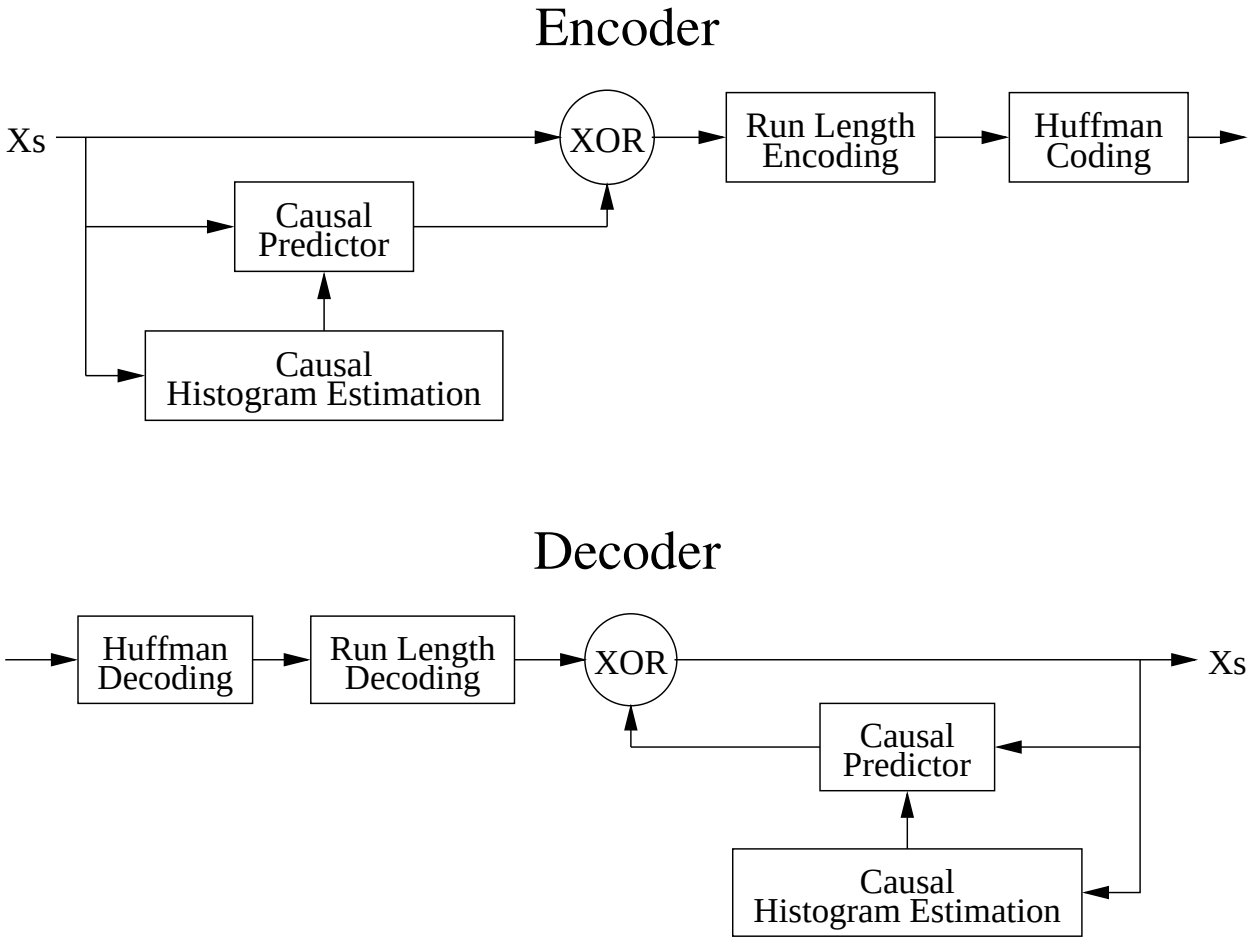
$$N(j) \leftarrow N(j) + 1$$

- Use updated values of $h(i,j)$ to compute new predictor at each pixel

$$\hat{p}(i|j) \leftarrow \frac{h(i,j)}{N(j)}$$

- Design decoder to track encoder

# Adaptive Predictive Entropy Coder Flow Diagram



# Lossy Source Coding

- Method for representing discrete-space signals with minimum distortion and bit-rate
- Outline
  - Rate-distortion theory
  - Karhunen-Loeve decorrelating Transform
  - Practical coder structures

# Distortion

- Let $X$ and $Z$ be random vectors in $\mathbb{R}^M$. Intuitively, $X$ is the original image/data and $Z$ is the decoded image/data.

  Assume we use the squared error distortion measure given by

  $$d(X,Y) = ||X - Z||^2$$

  Then the distortion is given by

  $$D = E\left[d(X,Y)\right] = E\left[||X - Z||^2\right]$$

- This actually applies to any quadratic norm error distortion measure since we can define

  $$\tilde{X} = AX \quad \text{and} \quad \tilde{Z} = AZ$$

  So

  $$\tilde{D} = E\left[\left\|\tilde{X} - \tilde{Z}\right\|^2\right] = E\left[||X - Z||_B^2\right]$$

  where $B = A^t A$.

# Lossy Source Coding: Theoretical Framework

- Notation for source coding

  $X_n \in I\!R^M$ for $0 \leq n < N$ - a sequence of i.i.d. random vectors

  $Y \in \{0,1\}^K$ - a $K$ bit random binary vector.

  $Z_n \in I\!R^M$ for $0 \leq n < N$ - the decoded sequence of random vectors.

  $$X^{(N)} = (X_0, \cdots, X_{N-1})$$
  $$Z^{(N)} = (Z_0, \cdots, Z_{N-1})$$

  – Encoder function: $Y = Q(X_0, \cdots, X_{N-1})$
  – Decoder function: $(Z_0, \cdots, Z_{N-1}) = f(Y)$

- Resulting quantities

  Bit-rate $= \frac{K}{N}$

  Distortion $= D_N(X^{(N)}, Z^{(N)}) = \frac{1}{N} \sum_{n=0}^{N-1} E\left[||X_n - Z_n||^2\right]$

- How do we choose $Y$ to minimize the bit-rate and distortion?

# Differential Entropy

- Notice that the information contained in a Gaussian random variable is infinite, so the conventional entropy $H(X)$ is not defined.

- Let $X$ be a random vector taking values in $\mathbb{R}^M$ with density function $p(x)$. Then we define the differential entropy of $X$ as

$$\begin{aligned} h(X) &= -\int_{x\in\mathbb{R}^M} p(x)\log_2 p(x)dx \\ &= -E\left[\log_2 p(X)\right] \end{aligned}$$

  $h(X)$ has units of bits

# Conditional Entropy and Mutual Information

- Let $X$ and $Y$ be a random vectors taking values in $\mathbb{R}^M$ with density function $p(x, y)$ and conditional density $p(x|y)$.
- Then we define the differential conditional entropy of $X$ given $Y$ as

$$\begin{aligned} h(X|Y) &= -\int_{x\in\mathbb{R}^M}\int_{x\in\mathbb{R}^M} p(x,y)\log_2 p(x|y) \\ &= -E\left[\log_2 p(X|Y)\right] \end{aligned}$$

- The mutual information between $X$ and $Y$ is given by

$$I(X;Y) = h(X) - h(X|Y) = I(Y;X)$$

- **Important:** The mutual information is well defined for both continuous and discrete random variables, and it represents the reduction in uncertainty of $X$ given $Y$.

# The Theoretical Rate

- We will define a theoretical quantity called "rate" which we will later show is the optimum bit-rate that can be achieved for a given distortion.

- For any $\delta > 0$, the rate is given by

$$R(\delta) = \inf_{Z} \{I(X_0; Z) : D \leq \delta\}$$

  where the infimum (i.e. minimum) over $Z$ is taken over all random variables $Z$.

- Properties of $R(\delta)$
  - $R(\delta)$ is a monotone decreasing function of $\delta$.
  - If $\delta \geq E\left[||X_0||^2\right]$, then $R(\delta) = 0$
  - $R(\delta)$ is a convex function of $\delta$

# Shannon's Source-Coding Theorem

- Shannon's Source-Coding Theorem:
  For any $R' > R(\delta)$ and $D' > \delta$ there exists a sufficiently large $N$ such that there is an encoder

$$Y = Q(X_0, \cdots, X_{N-1})$$

  which achieves

$$\frac{K}{N} \leq R'$$

  and

$$D_N(X^{(N)}, Z^{(N)}) \leq D' \ .$$

- Comments:
  - One can achieve a bit rate arbitrarily close to $R(\delta)$ at a distortion $D(\delta)$.
  - Proof is constructive (but not practical), and uses codes that are randomly distributed in the space $\mathbb{R}^{MN}$ of source symbols.