EE 637 Final

May 4, Spring 2011

**Name:** ______________________________________

**Instructions:**

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

## Good Luck.

# Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t - t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f - f_0)$$

$$x(t) y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f) Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy \\ f(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t - kT)$$

$$\text{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\text{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{rep}_{\frac{1}{T}}[X(f)]$$

$$\text{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \text{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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**Problem 1.**(20pt)

Consider an MRI that only images in one dimension, $x$. So for example, the object being imaged might be a thin rod oriented along the x-dimension.

In this example, assume that the magnetic field strength at each location is given by

$$M_o + G(t)x$$

where $M_o$ is the static magnetic field strength and $G(t)x$ is the linear gradient field in the $x$ dimension. Then the frequency of precession for a hydrogen atom (in rad/sec) is given by the product of $\gamma$, the gyromagnetic constant, and the magnetic field strength.

a) Calculate $\omega(x, t)$, the frequency of precession of a hydrogen atom at location $x$ and time $t$.

b) Calculate $\phi(x, t)$, the phase of precession of a hydrogen atom at location $x$ and time $t$ assuming that $\phi(x, 0) = 0$.

c) Calculate $r(x, t)$, the signal radiated from hydrogen atoms in the interval $[x, x + dx]$ at time $t$ in terms of $a(x)$, the quantity of precessing hydrogen atoms along the thin rod.

d) Calculate $r(t)$, the signal radiated from hydrogen atoms along the entire object.

e) Calculate an expression for $a(x)$ from the function $r(t)$.

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**Problem 2.**(20pt)

Let $T(m,n)$ be a 2-D random field of i.i.d. random variables which are uniformly distributed on the interval $[0,1]$. These thresholds are used to produce a binary halftone $B(m,n)$ for the gray level $g$, where $0 \leq g \leq 1$, using the relationship

$$B(m,n) = \begin{cases} 1 & \text{if } g \geq T(m,n) \\ 0 & \text{if } g < T(m,n) \end{cases}$$

a) Calculate $\mu$, the mean of $B(m,n)$.

b) Calculate $\sigma^2$, the variance of $B(m,n)$.

c) Calculate $R(k,l) = E\left[D(m,n)D(m+k,n+l)\right]$ where $D(m,n) = B(m,n) - \mu$, and calculate the power spectral density $S(e^{j\mu}, e^{j\nu})$ of $D(m,n)$.

d) Will $B(m,n)$ be a good quality halftone for the gray level $g$? Justify your answer.

e) Propose an alternative choice for $T(m,n)$ that will result in a better quality halftone.

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**Problem 3.**(20pt)

Below is a partial pseudo-code description of a subroutine called *ConnectedSet* for labeling all pixels connected to the pixel $s_0$. Let $c(s)$ be the set of connected neighbors to the pixel $s$, and let $Y$ be the image containing the label for each pixel.

```
Initialize Y_r = 0 for all r ∈ S
ClassLabel = 1
ConnectedSet(s_0, Y, ClassLabel) {
    B ← {s_0}
    While B is not empty {
        s ← any element of B
        ┌──────────────────────────────────┐
        │                                  │
        │                                  │
        │ ⟨Missing Pseudo-Code⟩            │
        └──────────────────────────────────┘
    }
    return(Y)
}
```

a) Fill in the missing section of code with the correct pseudo-code operations.

For the following problems, let $c(s) = \{r \in \partial s : X_r = X_s\}$ where $\partial s$ is the set of neighbors for the pixel $s$, and the image $X$ has the following specific form.

The image $X$

| | j | | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 |
| i | 0 | 1 | 1 | 2 | 2 | 2 |
| | 1 | 0 | 0 | 1 | 0 | 0 |
| | 2 | 0 | 1 | 1 | 0 | 0 |
| | 3 | 0 | 1 | 1 | 0 | 0 |
| | 4 | 0 | 1 | 0 | 0 | 1 |

b) Use the following table to specify the values of $Y$ returned by the subroutine when $s_0 = (i, j) = (0, 0)$ and a 4-point neighborhood is used.

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |

c) Use the following table to specify the values of $Y$ returned by the subroutine when $s_0 = (i, j) = (0, 0)$ and a 8-point neighborhood is used.

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |

d) Consider the case when $c(s) = \{r \in \partial s : |X_r - X_s| \leq T\}$. Explain the advantages and disadvantages of choosing $T$ to be large, and the advantages and disadvantages of choosing $T$ to be small.

e) Describe a practical strategy for selecting $T$ in a particular application. (Hint: Assume that you have some training data.)

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**Problem 4.**(20pt) Consider the following 2-D LSI systems. The first system has input $x(m,n)$ and output $y(m,n)$, and the second system has input $y(m,n)$ and output $z(m,n)$.

$$\begin{aligned} y(m,n) &= x(m,n) + ay(m,n-1) && \text{S1} \\ z(m,n) &= y(m,n) + bz(m-1,n) && \text{S2} \end{aligned}$$

a) Calculate the 2-D frequency response for the first system, $H_1(e^{j\mu}, e^{j\nu}) = \frac{Y(e^{j\mu},e^{j\nu})}{X(e^{j\mu},e^{j\nu})}$.

b) Calculate the 2-D frequency response for the second system, $H_2(e^{j\mu}, e^{j\nu}) = \frac{Z(e^{j\mu},e^{j\nu})}{Y(e^{j\mu},e^{j\nu})}$.

c) Calculate the 2-D frequency response for the combined system, $H_3(e^{j\mu}, e^{j\nu}) = \frac{Z(e^{j\mu},e^{j\nu})}{X(e^{j\mu},e^{j\nu})}$.

d) Calculate a single 2-D difference equation for the system $H_3(e^{j\mu}, e^{j\nu})$.

e) For what values of $a$ and $b$ is the 2-D system $H_3(e^{j\mu}, e^{j\nu})$ stable?

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**Problem 5.**(20pt) Consider focal plane array with detectors size $T \times T$. Let $g(x, y)$ denote the incoming light field, and let $s(m, n)$ denote the measurement from the $(m, n)^{th}$ detector. Then these are related by

$$s(m, n) = \int_{x=-T/2+mT}^{T/2+mT} \int_{y=-T/2+nT}^{T/2+nT} g(x, y)\, dx\, dy \ .$$

a) Calculate an expression for, $S(e^{j\mu}, e^{j\nu})$, the DSFT of $s(m, n)$ in terms of the function $G(u, v)$.

b) Give simple constraints on the bandwidth of the signal $g(x, y)$ that insure that the signal can be reconstructed (in theory) from its samples $s(m, n)$.

c) The sampled image is then displayed as

$$f(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(m, n) p(x - mT, y - nT) \ .$$

If the image is viewed on an LCD display, then give a reasonable choice of the function $p(x, y)$ to accurately model the display.

d) Assuming that $g(x, y)$ is appropriately band-limited, find an equation that directly relates $G(u, v)$ and $F(u, v)$ for the LCD display.

e) What discrete-space filter, $H(e^{j\mu}, e^{j\nu})$, should be applied to $s(m, n)$ in order to insure that $f(m, n) = g(m, n)$? Be specific.