Multivariate Gaussian distribution $x \in \mathbb{R}^p$

$$p(x) = \frac{1}{(2\pi)^{p/2}} |R|^{-1/2} \exp\left\{ x^t R^{-1} x \right\}$$

where R is a positive definite symmetric matrix.

Then

$$R = E \Lambda E^t$$

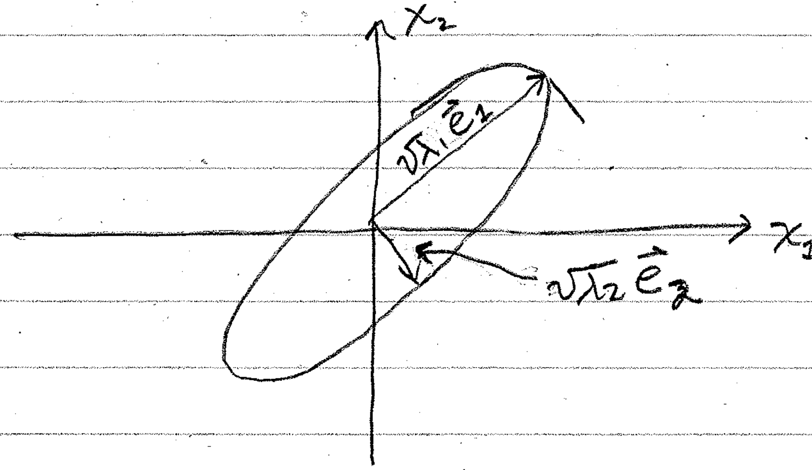
where $E = \begin{bmatrix} e_1 & e_K \end{bmatrix}$ is an orthonormal matrix of eigenvectors e_k and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ is a diagonal matrix of eigenvalues.

So we have $E^t E = I$

Intuitively, the function

$$f(x) = x^t R^{-1} x$$

has contour plots that look like



Consider $\tilde{X} = E^t X$ then

$$E\left[\tilde{X}\tilde{X}^t\right] = E\left[E^t X X^t E\right]$$

$$= E^t E\left[X X^t\right] E$$

$$= E^t R E$$

$$= E^t E \Lambda E^t E$$

$$= \Lambda$$

So the elements of $\tilde{X}$ are uncorrelated (independent) with variance $E[\tilde{X}_i^2] = \Lambda_{ii}$.

$$X \longrightarrow \boxed{E^t} \xrightarrow{\tilde{X}} \boxed{\Lambda^{-1/2}} \longrightarrow Z$$

$\quad$ white

$$E[ZZ^t] = I$$

So E^t decorrelates X

$\Lambda^{-1/2} E^t$ whitens X

$$E^t X = \begin{bmatrix} e_1^t \\ \vdots \\ e_p^t \end{bmatrix} \begin{bmatrix} X \end{bmatrix}$$

eigen signals

The e_k's are basis vectors to represent the signal X.

How do we determine the e_k's?

<u>Eigen image estimation</u>

How do we estimate R_x?

Assume we have n training vectors

$$X = [x_1, \ldots, x_n]$$

Then

$$R_x = E\left[X_k X_k^t\right] = \frac{1}{n} E\left[X X^t\right]$$

$$\cong \underbrace{\frac{1}{n} X X^t}_{} = S$$

This is the sample correlation matrix

$$S_{ij} = \frac{1}{n}\sum_{k=1}^{n} x_{ik}\, x_{jk}$$

$$S = E\, \Lambda\, E^t$$

estimate of the eigenvectors.

estimate of the eigenvalues.

Problem! How do we compute E ?

very large

<u>Singular Value Decomposition</u> (SVD)

For $n<p$ it looks like:

$$\underset{p\times n}{X} = \underset{p\times n}{U}\;\underset{n\times n}{\Sigma}\;\underset{n\times n}{V^t}$$

$U \Leftarrow$ Left hand singular vectors
columns are orthonormal

$V \Leftarrow$ right hand singular vectors
columns are orthonormal

$\Sigma \Leftarrow$ diagonal matrix of
singular values.

Notice that

$$XX^t = U \Sigma V^t V \Sigma U^t$$

$$= U \Sigma^2 U^t$$

So U is the set of desired
eigen vectors of XX^t.

How to compute U?

Notice $X^t X = \underbrace{V \Sigma^2 V^t}_{n \times n}$

$V \Leftarrow$ eigenvectors of **much**
smaller matrix

Algorithm

1) Find eigenvectors V of the matrix X^*X

2) Compute $XV = U\Sigma$

Then the columns of U are the normalized columns of XV, and Σ are the normalization factors.