EE 637 Final

May 4, Spring 2010

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Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 20 points for a total score of 100 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi ft_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)Y^*(f)df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \\ f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Rep and Comb relations

$$\text{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\text{comb}_T\left[x(t)\right] = x(t)\sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\text{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\text{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\text{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\text{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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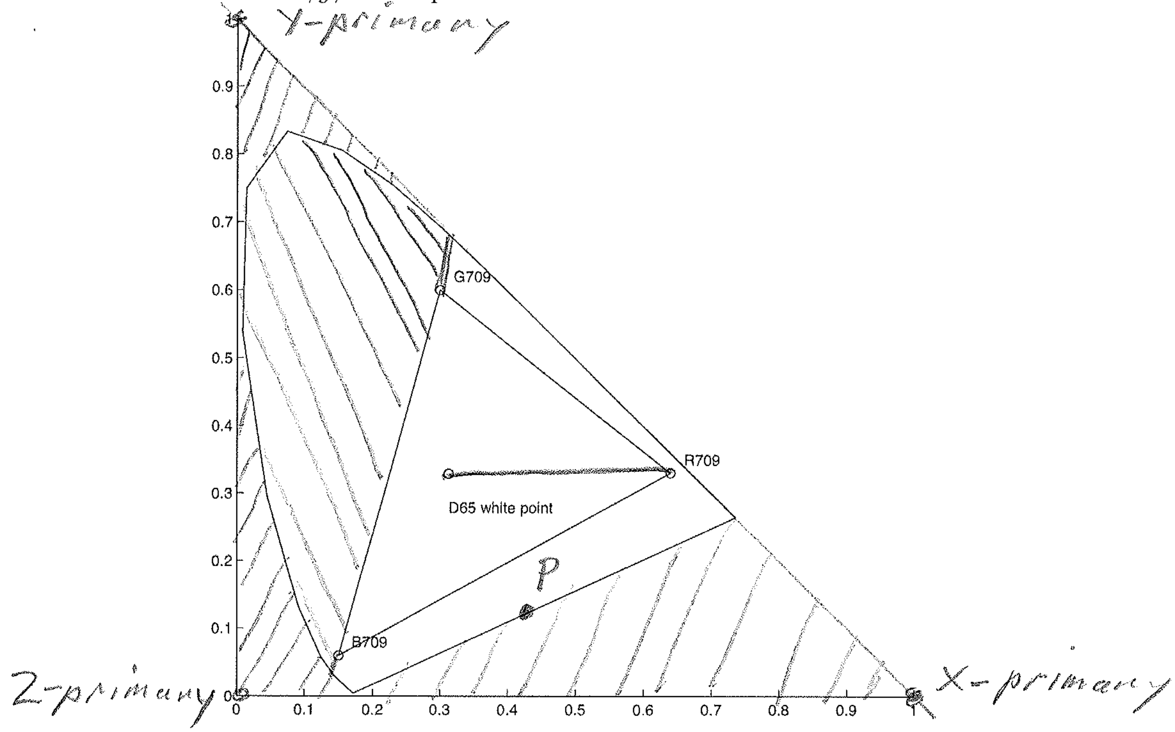
Problem 1.(20pt)

Consider the standard chromaticity diagram below, and assume that you are using a display device with standard 709 r, g, b color primaries.



a) Draw a triangle corresponding to all colors with positive values of X, Y, and Z. Label the three primaries for this triangle as "X-primary", "Y-primary", and "Z-primary".

b) Label the region of the chromaticity diagram corresponding to imaginary colors. (Use 45 deg diagonal hash marks to indicate this region of the diagram.)

c) Label ALL real colors with $r < 0$ on the chromaticity diagram. (Use -45 deg diagonal hash marks to indicate this region of the diagram.)

d) Place a point on the diagram corresponding to a highly saturated color that is NOT formed by a single wavelength of light. Label this point with the letter "P".

e) Draw a line on the plot corresponding to all color that can be formed with a combination of the D65 white and R709.

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Problem 2.(20pt)

The approximate *Lab* color space transform is given by

$$
\begin{aligned}
L &= 100(Y/Y_0)^{1/3} \\
a &= 500\left[(X/X_0)^{1/3} - (Y/Y_0)^{1/3}\right] \\
b &= 200\left[(Y/Y_0)^{1/3} - (Z/Z_0)^{1/3}\right]
\end{aligned}
$$

a) Qualitatively specify the colors corresponding to the following values of L, a, b: 1) $L = 100$, $a = 0$, $b = 0$; 2) $L = 100$, $a =$ large positive, $b = 0$; 3) $L = 100$, $a = 0$, $b =$ large positive; 4) $L = 100$, $a =$ large negative, $b = 0$; and 5) $L = 100$, $a = 0$, $b =$ large negative;

b) Which component of the L, a, b coordinate system requires the greatest spatial frequency to accurately represent? Why?

c) Is the L, a, b coordinate system suitable for representing images that will be JPEG compressed? Why or why not?

d) Imagine that an image with large energy in high frequencies is viewed from a great distance, and you would like to know the average color that the viewer sees.

Should you low pass filter the L, a, b image to determine the average color? Either justify that this approach is correct, or explain a better approach.

a) $(100, 0, 0) \Rightarrow$ Bright white corresponding to (X_0, Y_0, Z_0)

(100, large pos, 0 $\Rightarrow$ saturated red. (opponent to green).

(100, 0, large pos) $\Rightarrow$ saturated yellow (opponent to blue)

(100, large neg, 0) $\Rightarrow$ saturated green (opponent color to red)

(100, 0, large neg) $\Rightarrow$ saturated Blue (opponent color to yellow)

Answer 2a)

For (L,a,b) = (100, a, 0) then it must be that $Y=Y_0$ and $Z=Z_0$. So the color's (X,Y,Z) components must be given by $(X,Y,Z) = (X_0, Y_0, Z_0) + (X, 0, 0)$ for some value of X. This means that the chromaticity of the color (X,Y,Z) must fall on a line connecting **W** = (X_0, Y_0, Z_0) and **X** = (X, 0, 0) where **W** is the white point of the L,a,b transform and **X** is the imaginary primary for the X component of the (X,Y,Z) coordinate system.

This line between **W** and **X** is illustrated in the figure below. The resulting color is can then be expressed as Color = $\mathbf{W} + \alpha\mathbf{X}$ where α is a constant. When a=0, then Color = **W**, and it must be that $\alpha=0$. When a>0, then Color = $\mathbf{W} + \alpha\mathbf{X}$, where $\alpha>0$; and when a<0, then Color = $\mathbf{W} + \alpha\mathbf{X}$, where $\alpha<0$. The largest value of a that results in a real color is indicated by the point **R** on the diagram, and the smallest value of a that results in a real color is indicated by the point **G** on the diagram.

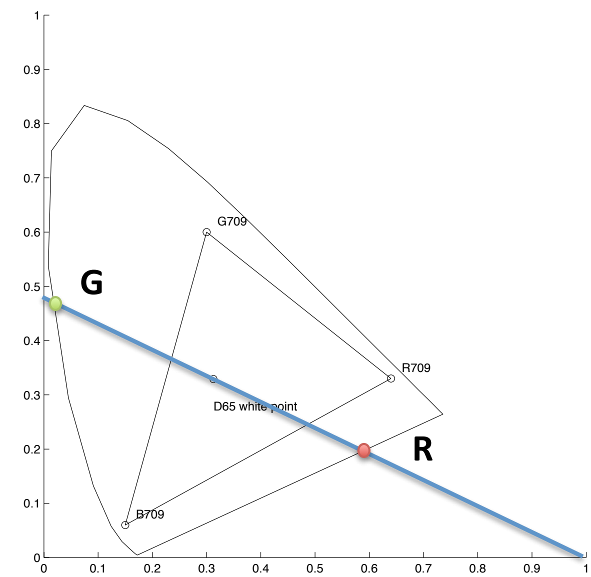

Figure 1 – Line corresponding to L=100, b=0.

For (L,a,b) = (100, 0, b) the situation is similar, as illustrated in Figure 3. In this case, the chromaticity of the color (X,Y,Z) must fall on a line connecting **W** = (X_0, Y_0, Z_0) and **Z** = (0, 0, Z) where **W** is the white point of the L,a,b transform and **Z** is the imaginary primary for the Z component of the (X,Y,Z) coordinate system.

This line between **W** and **Z** is illustrated in the Figure 3. The resulting color can then be expressed as Color = $\mathbf{W} + \beta\mathbf{Z}$ where β is a constant. When b=0, then Color = **W**, and it must be that $\beta=0$. When b<0, then Color = $\mathbf{W} + \beta\mathbf{Z}$, where $\beta>0$; and when b>0, then Color = $\mathbf{W} + \beta\mathbf{Z}$, where $\beta<0$. The largest value of b that results in a real color is indicated by the point **Y** on the diagram, and the smallest value of b that results in a real color is indicated by the point **B** on the diagram.

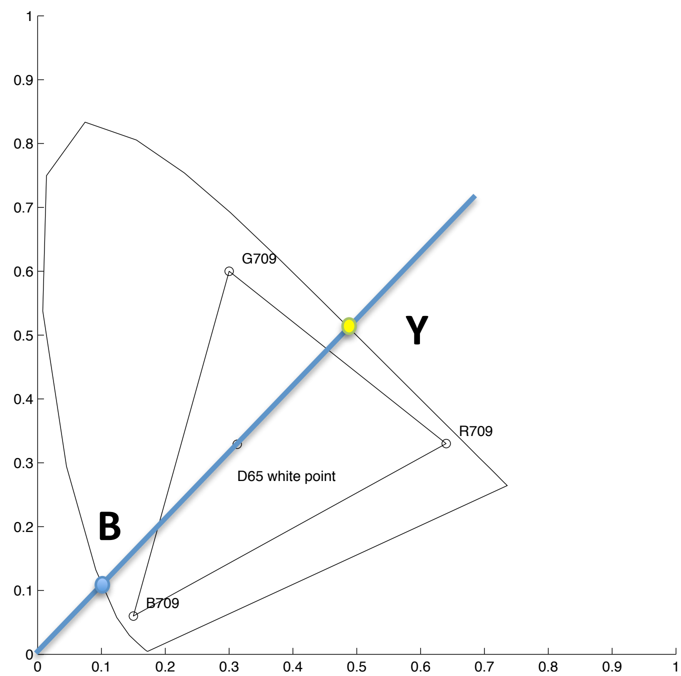

Figure 2 – Line corresponding to L=100, a=0.

Referring to Figure 2, we can see that **R**, **G**, **B**, and **Y** correspond to the actual colors of red, green, blue, and yellow.

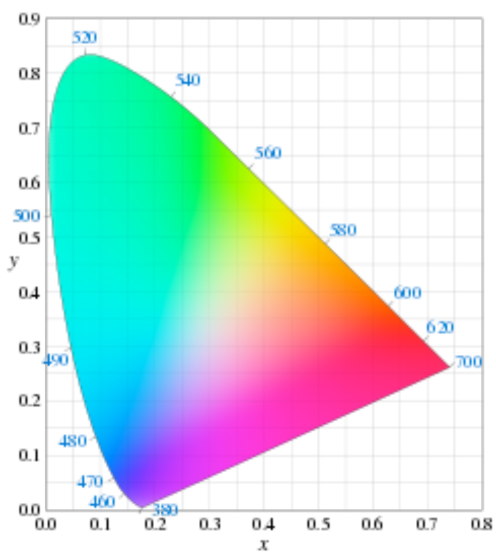

Figure 2 – Colors of chromaticity diagram.

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b) L requires greatest spatial frequency because the contrast sensitivity falls off most slowly in luminance.

c) No because it is designed for color matching of large patches corresponding to low spatial frequency.

d) No, lowpass filtering (L, a, b) will produce the wrong result. The correct approach is to convert to (X, Y, Z), then low pass filter, then convert back to Lab.

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Problem 3.(20pt)

Assume that for achromatic images, you use the following fidelity metric for the human visual system,

$$D = \sum_{m,n} \left(h(m,n) * b(m,n) - h(m,n) * g(m,n)\right)^2$$

where D is a measure of distortion between the linear gray scale image $g(m,n)$ and the binary image $b(m,n)$, and $*$ indicates 2D convolution.

Furthermore, assume that

$$h(m,n) = [\delta(m,n) + (1/2)\left(\delta(m-1,n) + \delta(m+1,n)\right)]*[\delta(m,n) + (1/2)\left(\delta(m,n-1) + \delta(m,n+1)\right)]$$

where $*$ represents 2D convolution.

a) Calculate the DSFT, $H(e^{j\mu}, e^{j\nu})$ of $h(m,n)$, and sketch its shape.

b) What are the values of $H(e^{j0}, e^{j0})$, $H(e^{j\pi}, e^{j0})$, $H(e^{-j\pi}, e^{j0})$, $H(e^{j0}, e^{j\pi})$, and $H(e^{j0}, e^{-j\pi})$.

c) Assuming your objective is to represent the gray scale image $g(m,n)$ by the binary image $b(m,n)$, then is it best for $d(m,n) = b(m,n) - g(m,n)$ to contain mostly high frequencies or low frequencies? Why?

d) If $g(m,n) = 1/2$, then determine all binary patterns $b(m,n)$ (i.e. a pattern of 1's and 0's) that best matches $g(m,n)$.

e) How could the distortion measure, D, be improved to better account for contrast?

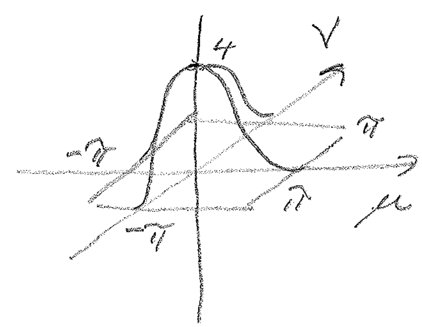
a) $\delta(n) + \frac{1}{2}\left(\delta(n-1) + \delta(n+1)\right)$

$\overset{DTFT}{\Longleftrightarrow} \quad 1 + \frac{1}{2}\left(e^{-j\omega} + e^{j\omega}\right)$

$= 1 + \cos(\omega)$

$h(m,n) \overset{DSFT}{\Longleftrightarrow} \left(1+\cos(\mu)\right)\left(1+\cos(\nu)\right)$

$= H(e^{j\mu}, e^{j\nu})$



b) $H(e^{j0}, e^{j0}) = 4$

$H(e^{j\pi}, e^{j0}) = H(e^{-j\pi}, e^{j0})$

$= 0\,(2) = 0$

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$$H(e^{j0}, e^{j\pi}) = H(e^{j0}, e^{j\pi}) = 2(0) = 0$$

c) It is best to push error to high frequencies, where they are more difficult to see.

d) The following binary pattern result in no low frequency error, and produce $D = 0$.

$$b(m,n) = \frac{(-1)^m(-1)^n + 1}{2}$$

or

$$b(m,n) = \frac{(-1)^m + 1}{2}$$

or

$$b(m,n) = \frac{(-1)^n + 1}{2}$$

e)

$$D' = \sum_{m,n} \left((h(m,n) * b(m,n))^{1/3} - (h(m,n) * g(m,n))^{1/3} \right)^2$$

The power $1/3$ accounts for sensitivity to contrast.

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Problem 4.(20pt)

Consider a sequence of N i.i.d. random variables, X_n, each with density

$$p(x|\mu) = \frac{1}{z}\exp\left(-\rho(x-\mu)\right)$$

where μ is a parameter of the distribution, and z is a normalizing constant given by

$$z = \int_{\Re} \exp\left(-\rho(x)\right) dx \ .$$

Then the maximum likelihood estimate of μ is defined as

$$\begin{aligned} \hat{\mu} &= \arg\max_{\mu} \{p(x_1,\cdots,x_N|\mu)\} \\ &= \arg\max_{\mu} \{\log p(x_1,\cdots,x_N|\mu)\} \end{aligned}$$

where $p(x_1,\cdots,x_N|\mu)$ is the joint density for the sequence of random variables $(X_1,\cdots,X_N)$.

a) Derive an expressions for the joint density, $p(x_1,\cdots,x_N|\mu)$, and $\log p(x_1,\cdots,x_N|\mu)$.

b) Derive a general expression for the maximum likelihood estimate of μ.

c) Calculate the maximum likelihood estimate of μ when $\rho(x) = x^2$.

d) Calculate the maximum likelihood estimate of μ when $\rho(x) = |x|$.

e) What is the advantage of using $\rho(x) = |x|$ rather than $\rho(x) = x^2$?

a) $p(x_1 \ldots x_N|\mu) = \frac{1}{Z^N} \prod_{i=1}^{N} \exp\{-\rho(x_i-\mu)\}$

$= \frac{1}{Z^N} \exp\{-\sum_{i=1}^{N} \rho(x_i-\mu)\}$

$\log p(x_1, \ldots, x_N|\mu) = -\sum_{i=1}^{N} \rho(x_i-\mu) - N\log(Z)$

b)

$\hat{\mu} = \arg\min_{\mu} \sum_{i=1}^{N} \rho(x_i-\mu)$

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c) $$\hat{\mu} = \arg\min_{\mu} \sum_{i=1}^{N} (x_i - \mu)^2$$

$$\frac{d}{d\mu} \sum_{i=1}^{N} (x_i - \mu)^2 = 0$$

$$\sum_{i=1}^{N} 2(x_i - \hat{\mu})(-1) = 0$$

$$\sum_{i=1}^{N} x_i - N\hat{\mu} = 0 \Rightarrow \hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

d) $$\frac{d}{d\mu} \sum_{i=1}^{N} |x_i - \mu| = 0$$

$$\sum_{i=1}^{N} \text{sign}(x_i - \mu) = 0 \Leftarrow \begin{array}{c} \#\text{ of } x_i > \mu \\ = \\ \#\text{ of } x_i < \mu \end{array}$$

$$\mu = \text{median}(x_1, \ldots, x_N)$$

e) The function $\rho(x) = |x|$ results in an ML estimate that is less sensitive to outliers, or equivalently more robust.

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Problem 5.(20pt)

Let $X = [x_1, \cdots, x_N]$ be a $P \times N$ matrix formed by P dimensional column vectors, $x_n \in \Re^P$ where $N < P$. We will assume that each column vector is an independent multivariate Gaussian random vector with distribution $N(0, R)$, where R is a positive definite and symmetric matrix.

Furthermore, let $X = U\Sigma V^t$ be the singular value decomposition of X, where $\Sigma_{i,i} \geq \Sigma_{j,j}$ when $i < j$.

a) Write a simple matrix expression for the sample covariance, $\hat{R}$.

b) Let $\hat{R} = E\Lambda E^t$ be the eigen decomposition of the sample covariance matrix, where E is the orthonormal transform with eigenvectors as columns and Λ is the diagonal matrix of eigen values. (Without loss of generality assume that the eigenvalues are ordered from largest to smallest so that $\Lambda_{i,i} \geq \Lambda_{j,j}$ when $i < j$.

How many non-zero eigenvalues does the matrix $\hat{R}$ contain?

c) Specify the eigenvectors and eigenvalues of $\hat{R}$ in terms of the SVD of X.

d) In some application, you can only use two numbers to specify each vector, x_n. So each vector must be approximated by

$$x_n \approx a_n e_1 + b_n e_2$$

where $e_1 \in \Re^P$ and $e_2 \in \Re^P$ are two orthonormal vectors, and a_n and b_n are two scalar values used to specify each vector, x_n.

What is the best choice of e_1 and e_2?

a) $\hat{R} = \frac{1}{N} X X^t$

b) N non-zero eigenvalues.

c) $\hat{R} = \frac{1}{N} X X^t = \frac{1}{N} U \Sigma V^t (U \Sigma V^t)^t$

$= \frac{1}{N} U \Sigma V^t V \Sigma U^t$

$= \frac{1}{N} U \Sigma^2 U^t$

So $E = U$ and $\Lambda = \frac{1}{N} \Sigma^2$

d) e_1 and e_2 should be the first and second columns of U, so that e_1 is the singular vector corresponding to the <u>largest</u> singular value Σ_{11}, and e_2 is the singular

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Vector corresponding to the

second largest singular value Σ_{22}.