# Image Restortation

- Problem:

  – You want to know some image $X$.

  – But you only have a corrupted version $Y$.

  – How do you determine $X$ from $Y$?

- Corruption may result from:

  – Additive noise

  – Nonadditive noise

  – Linear distortion

  – Nonlinear distortion

# Optimum Linear FIR Filter

- Find an "optimum" linear filter to compute $X$ from $Y$.

- Filter uses input window of $Y$ to estimate each output pixel $X_s$.

- Filter can be designed to be minimize mean squared error (MMSE).

- The estimate of $X_s$ is denoted by $\hat{X}_s$.

- $W(s)$ denotes the window about $s$.

- The estimate, $\hat{X}_s$, is a function of $Y_{W(s)}$.

# Application of Optimum Filter

$$\hat{X}_s = f(Y_{W(s)})$$



Measured Image Y                    Restored Image $\hat{X}$(Y)
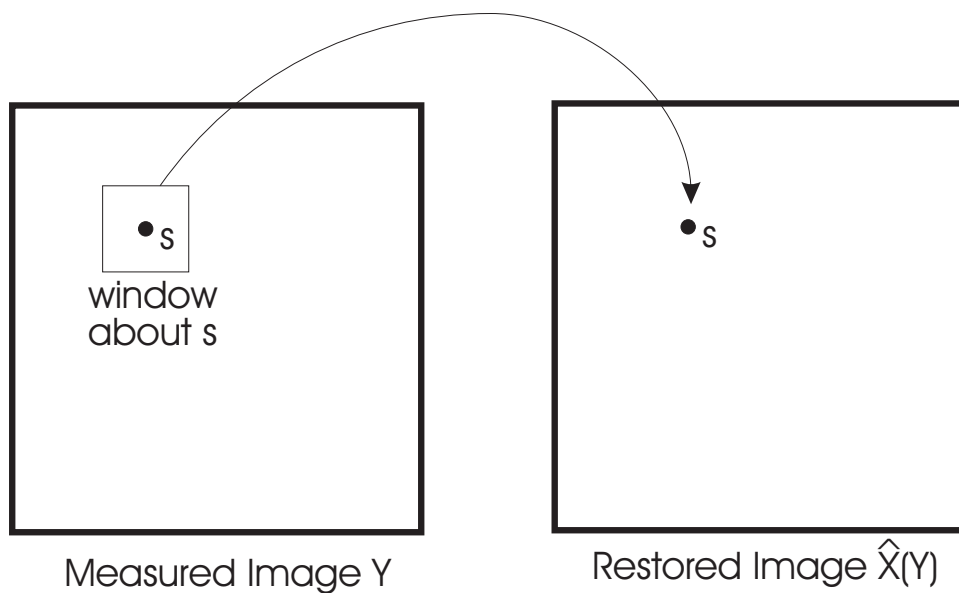
- The function $f(Y_{W(s)})$ is designed to produce a MMSE estimate of $X$.

- If $f(Y_{W(s)})$ is:

    - Linear $\Rightarrow$ linear space invariant filter.
    - Nonlinear $\Rightarrow$ nonlinear space invariant filter.

- This filter can reduce the effects of all types of corruption.

# Optimality Properities of Linear Filter

- If both images are jointly Gaussian:

  - Then MMSE filter is linear.

$$\hat{X}_s = E[X_s | Y_{W(s)}]$$

$$= \mathbf{A}\, Y_{W(s)} + b$$

- If images are not jointly Gaussian:

  - Then MMSE filter is generally not linear.

$$\hat{X}_s = E[X_s | Y_{W(s)}]$$

$$= f(Y_{W(s)})$$

  - However, the MMSE linear filter can still be very effective!

# Formulation of MMSE Linear Filter: Definitions

- $W(s)$ - window about the pixel $s$.

- $p$ - number of pixels in $W(s)$

- $z_s$ - row vector containing pixels of $Y_{W(s)}$.

- $\theta$ - column containing filter parameters

- Detailed definitions:

  - Definition of $W(s)$

  $$W(s) = [s, s + r_1, \ldots, s + r_{p-1}]$$

  where $r_1, \ldots, r_{p-1}$ index neighbors.
  - Definition of $z_s$

  $$z_s = [y_s, y_{s+r_1}, \ldots, y_{s+r_{p-1}}]$$

  - Definition of $\theta$

  $$\theta = [\theta_0, \ldots, \theta_{p-1}]$$

# Formulation of MMSE Linear Filter: Objectives

- Linear filter is given by

$$\hat{x}_s = z_s \theta$$

- Mean squared error is given by

$$
\begin{aligned}
MSE &= E[|x_s - \hat{x}_s|^2] \\
&= E[|x_s - z_s \theta|^2]
\end{aligned}
$$

- The MMSE filter parameters $\theta^*$ are given by

$$\theta^* = \arg \min_\theta E[|x_s - z_s \theta|^2] \ .$$

- How do we solve this problem?

# More Matrix Notation

- Define the subset $S_0$ of image pixels.

  1. $S_0 \subset S$

  2. $S_0$ contains $N_0 < N$ pixels

  3. $S_0$ usually does not contain pixels on the boundary of the image.

  4. $S_0 = [s_1, \ldots, s_{N_0}]$

- Define the $N_0 \times p$ matrix $Z$

$$Z = \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix}.$$

- Define the $N_0 \times 1$ column vectors $X$ and $\hat{X}$

$$X = \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \text{ and } \hat{X} = \begin{bmatrix} \hat{x}_{s_1} \\ \hat{x}_{s_2} \\ \vdots \\ \hat{x}_{s_{N_0}} \end{bmatrix}.$$

- Then

$$X \approx \hat{X} = Z\theta$$

# Least Squares Linear Filter

- We expect that

$$
\begin{aligned}
MSE &= E[|x_s - z_s\theta|^2] \\
&\approx \frac{1}{N_0} \sum_{s \in S_0} |x_s - z_s\theta|^2 \\
&= \frac{1}{N_0} ||X - Z\theta||^2
\end{aligned}
$$

- So we may solve the equation

$$
\theta^* = \arg\min_{\theta} ||X - Z\theta||^2
$$

- The solution $\theta^*$ is the least squares estimate, of $\theta$, and the estimate

$$
\hat{X} = Z\theta^*
$$

is known as the least squares filter.

# Computing Least Squares Linear Filter

$$\theta^* = \arg\min_\theta \frac{1}{N_0} ||X - Z\theta||^2$$

- So

$$
\begin{aligned}
\theta^* &= \arg\min_\theta \left( \frac{1}{N_0} ||X - Z\theta||^2 \right) \\
&= \arg\min_\theta \left( \frac{1}{N_0} (X - Z\theta)^t (X - Z\theta) \right) \\
&= \arg\min_\theta \left( \frac{1}{N_0} (X^t X - 2\theta^t Z^t X + \theta^t Z^t Z\theta) \right) \\
&= \arg\min_\theta \left( \frac{X^t X}{N_0} - 2\theta^t \frac{Z^t X}{N_0} + \theta^t \frac{Z^t Z}{N_0}\theta \right) \\
&= \arg\min_\theta \left( \theta^t \frac{Z^t Z}{N_0}\theta - 2\theta^t \frac{Z^t X}{N_0} \right)
\end{aligned}
$$

# Covariance Estimates

- Define the $p \times p$ matrix

$$\hat{R}_{zz} \triangleq \frac{Z^t Z}{N_0}$$

$$= \frac{1}{N_0} \left[ z_{s_1}^t, z_{s_2}^t, \ldots, z_{s_{N_0}}^t \right] \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix}$$

$$= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t z_{s_i}$$

- Define the $p \times 1$ vector

$$\hat{r}_{zx} \triangleq \frac{Z^t X}{N_0}$$

$$= \frac{1}{N_0} \left[ z_{s_1}^t, z_{s_2}^t, \ldots, z_{s_{N_0}}^t \right] \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix}$$

$$= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t x_{s_i}$$

- So

$$\theta^* = \arg \min_{\theta} \left( \theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

# Interpretation of $\hat{R}_{zz}$ and $\hat{r}_{zx}$

- $\hat{R}_{zz}$ is an estimate of the covariance of $z_s$.

$$
\begin{aligned}
E\left[\hat{R}_{zz}\right] &= E\left[\frac{1}{N_0} \sum_{s=1}^{N} z_s^t z_s\right] \\
&= E[z_s^t z_s] \\
&= R_{zz}
\end{aligned}
$$

- $\hat{r}_{zx}$ is an estimate of the cross correlation between $z_s$ and $x_s$.

$$
\begin{aligned}
E\left[\hat{r}_{zx}\right] &= E\left[\frac{1}{N_0} \sum_{s=1}^{N} z_s^t x_s\right] \\
&= E[z_s^t x_s] \\
&= r_{zx}
\end{aligned}
$$

# Solution to Least Squares Linear Filter

- We need
$$\theta^* = \arg\min_\theta \frac{1}{N_0}||X - Z\theta||^2$$
We have shown this is equivalent to
$$\theta^* = \arg\min_\theta \left(\theta^t \hat{R}_{zz}\theta - 2\theta^t \hat{r}_{zx}\right)$$

- Taking the gradient of the cost functional
$$0 = \left.\bigtriangledown_\theta \left(\theta^t \hat{R}_{zz}\theta - 2\theta^t \hat{r}_{zx}\right)\right|_{\theta=\theta^*}$$
$$= \left.\left(2\hat{R}_{zz}\theta - 2\hat{r}_{zx}\right)\right|_{\theta=\theta^*}$$
Solving for $\theta^*$ yeilds
$$\theta^* = \left(\hat{R}_{zz}\right)^{-1} \hat{r}_{zx}$$

# Summary of Solution to Least Squares Linear Filter

- First compute

$$\hat{R}_{zz} \;=\; \frac{1}{N_0} \sum_{s=1}^{N} z_s^t z_s$$

$$\hat{r}_{zx} \;=\; \frac{1}{N_0} \sum_{s=1}^{N} z_s^t x_s$$

- Then compute

$$\theta^* = \left( \hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

- The vector $\theta^*$ then contains the values of the filter coefficients.

# Training

- $\theta^*$ is usually estimated from "training" data.

- Training data

  - Generally consists of image pairs $(X, Y)$ where $Y$ is the measured data and $X$ is the undistorted image.
  - Should be typical of what you might expect.
  - Can often be difficult to obtain.

- Testing data

  - Also consists of image pairs $(X, Y)$.
  - Is used to evaluate the effectiveness of the filters.
  - Should never be taken from the training data set.

- Training versus Testing

  - Performance on training data is always better than performance on testing data.
  - As the amount of training data increases, the performance on training and testing data both approach the best achievable performance.

# Comments

- Wiener filter is the MMSE **linear** filter.

- Wiener filter may be optimal, but it isn't always good.

  - Linear filters blur edges
  - Linear filters work poorly with non-Gaussian noise.

- Nonlinear filters can be designed using the same methodologies.

# Is MMSE a Good Quality Criteria for Images?

- In general, NO! ... But sometimes it is OK.

- For achromatic images, it is best to choose $X$ and $Y$ in $L^*$ or gamma corrected coordinates.

- Let $H$ be a filter that implements the CSF for the human visual system.

  - Then a better metric of error is

$$
\begin{aligned}
HVSE &= ||H(X - \hat{X})||^2 \\
&= \left(X - \hat{X}\right)^t H^t H \left(X - \hat{X}\right) \\
&= ||X - \hat{X}||_B^2
\end{aligned}
$$

  where $B = H^t H$.
  - $||X - \hat{X}||_B^2$ is a quadratic norm.

- What is the minimum HVSE estimate $\hat{X}$?

# Answer

- The answer is $\hat{X} = E[X|Y]$.

    - This is the same as for mean squared error!
    - The conditional expectation minimizes any quadratic norm of the error.
    - This is also true for non-Gaussian images.

- Let $\hat{X} = AY_{W(s)} + b$ be the MMSE **linear** filter.

    - This filter is also the minimum HVSE **linear** filter.
    - This is also true for non-Gaussian images.

# Proof

- Define $V \stackrel{\triangle}{=} HX$ and $B = H^t H$

$$\min_{\hat{X}} E\left[||X - \hat{X}||_B^2\right]$$

$$= \min_{\hat{X}} E\left[||H(X - \hat{X})||^2\right]$$

$$= \min_{\hat{V}} E\left[||V - \hat{V}||^2\right]$$

$$= E\left[||V - E[V|Y]||^2\right]$$

$$= E\left[||HX - E[HX|Y]||^2\right]$$

$$= E\left[||H(X - E[X|Y])||^2\right]$$

$$= E\left[||X - E[X|Y]||_B^2\right]$$

- So, $\hat{X} = E[X|Y]$ minimizes the error measure.

$$HVSE = ||X - \hat{X}||_B^2 \ .$$