EE 637 Final

May 5, Spring 2009

**Name:** ________________________________________

**Instructions:**

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 40 points for a total score of 200 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

## Good Luck.

# Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f)e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t)e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)Y^*(f)df$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(f)$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy \\ f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

- Rep and Comb relations

$$\text{rep}_T\left[x(t)\right] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\text{comb}_T\left[x(t)\right] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\text{comb}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\text{rep}_{\frac{1}{T}}\left[X(f)\right]$$

$$\text{rep}_T\left[x(t)\right] \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\text{comb}_{\frac{1}{T}}\left[X(f)\right]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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**Problem 1.**(40pt)

Derive each of the following properties.

a) Show that if $g(t)$ has a CTFT of $G(f)$, then $g(t-a)$ has a CTFT of $e^{-2\pi jaf}G(f)$.

b) Show that if $g(t)$ has a CTFT of $G(f)$, then $g(t/a)$ has a CTFT of $|a|G(af)$.

c) Show that if $x_n$ has a DTFT of $X(e^{j\omega})$, then $(-1)^n x_n$ has a DTFT of $X(e^{j(\omega-\pi)})$.

d) Show that if $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ has a CSFT of $G\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$, then $g\left(A\begin{bmatrix} x \\ y \end{bmatrix}\right)$ has a CSFT of $|A|^{-1}G\left(\left(A^{-1}\right)^t\begin{bmatrix} u \\ v \end{bmatrix}\right)$.

(Hint: Use the notation $r = \begin{bmatrix} x \\ y \end{bmatrix}$ and $f = \begin{bmatrix} u \\ v \end{bmatrix}$, so that $G(f) = \int_{\Re^2} g(r)e^{-jr^tf}dr$.)

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**Problem 2.**(40pt)

Consider a color imaging device that takes input values of $(r, g, b)$ and produces output $(X, Y, Z)$ values given by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = A \begin{bmatrix} r^\alpha \\ g^\alpha \\ b^\alpha \end{bmatrix} .$$

where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} .$$

a) Calculate the white point of the device in chromaticity coordinates.

b) What are the primaries associated with the $r$, $g$, and $b$ components respectively? Again, use chromaticity coordinates to specify your answer.

c) What is the gamma of the device?

d) Calculate the values of $(r, g, b)$ that will produce the color of an equal energy white. (Hint: You can express your solution in terms of $A^{-1}$.)

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**Problem 3.**(40pt)

Let $Y \in \Re^N$ be a vector containing the pixels in an image window. We can model $Y$ as

$$Y = tS + W$$

where $t \in \Re^N$ is a deterministic column vector of length $N$, $S$ is scalar valued Gaussian random variable with mean 0 and variance $\sigma^2$, and $W$ is a independent Gaussian random vector of correlated noise with distribution $N(0, R_w)$ where $R_w$ is an $N \times N$ positive definite covariance matrix.

Intuitively, $Y$ is composed of a signal $tS$ obscured by noise $W$. Our objective is to estimate the value of $S$ from the observations $Y$. To do this, we will form a MMSE linear estimator for $S$ given by

$$\hat{S} = Y^t \theta$$

where $\theta \in \Re^N$ is a vector of coefficients.

Furthermore, define the covariance matrix of $Y$ given by

$$R_y = E\left[ YY^t \right] \ ,$$

and the cross-covariance column vector of $Y$ and $S$ given by

$$b = E\left[ YS \right] \ .$$

a) Calculate an expression for the MSE given by $E\left[ ||S - \hat{S}||^2 \right]$ in terms of $R_y$, $b$, $\sigma^2$, and $\theta$.

b) Use the expression from part a) to compute the value of $\theta$ that produces the MMSE estimate of $S$.

c) Calculate $R_y$ in terms of $t$, $\sigma^2$, and $R_w$.

d) Calculate $b$ in terms of $t$, $\sigma^2$, and $R_w$.

e) Use the above results to calculate a closed form expression for $\hat{S}$.

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**Problem 4.**(40pt)

Consider a non-linear prediction problem for which we are trying to predict the value of a scalar $Y_n$ from a vector of observations $Z_n$. Our assumption is that we can estimate $Y_n$ using the non-linear predictor given by

$$\hat{Y}_n = f(Z_n, \theta)$$

where $\theta \in \Re^p$ is a $p$ dimensional parameter vector that controls the behavior of the nonlinear predictor.

Fortunately, we are given some training data pairs with the form $(Y_n, Z_n)$.[1] The data is partitioned into two sets. The first set, $n \in S_1$, contains $N = |S_1|$ pairs, and is used for training purposes. The second set, $n \in S_2$, contains $M = |S_2|$ pairs, and is used for testing purposes.

Using these data, we can define the training MSE as

$$MSE_1(\theta) = \frac{1}{N} \sum_{n \in S_1} \| Y_n - f(Z_n, \theta) \|^2 ,$$

the testing MSE as

$$MSE_2(\theta) = \frac{1}{M} \sum_{n \in S_2} \| Y_n - f(Z_n, \theta) \|^2 ,$$

and the expected MSE as

$$MSE_3(\theta) = E\left[ \| Y_n - f(Z_n, \theta) \|^2 \right] .$$

Based on these error measures, we can define the following two estimates for the parameter vector.

$$\hat{\theta} = \arg\min_{\theta} MSE_1(\theta)$$

$$\theta^* = \arg\min_{\theta} MSE_3(\theta)$$

a) Which of the two quantities would you expect to be smaller, $MSE_2(\hat{\theta})$ or $MSE_2(\theta^*)$? Why?

b) What is the disadvantage of using $MSE_2(\theta^*)$?

c) Approximately how large should $N$ be in order for $\hat{\theta}$ to be useful?

d) Sketch the plots of $MSE_1(\hat{\theta})$, $MSE_2(\hat{\theta})$, and $MSE_2(\theta^*)$ as a function of the amount of training data $N$.

e) Which value would you expect to be smaller, $MSE_1(\hat{\theta})$ or $MSE_2(\hat{\theta})$. Why?

f) If you are reporting results of your experiment, which value should you report, $MSE_1(\hat{\theta})$ or $MSE_2(\hat{\theta})$. Why?

[1] Assume that each training data pair is independent, and each pairs has the same distribution.

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**Problem 5.**(40pt)

Consider the 1-D error diffusion algorithm specified by the equations

$$\begin{aligned} b_n &= Q(y_n) \\ e_n &= y_n - b_n \\ y_n &= x_n + e_{n-1} \end{aligned}$$

where $x_n$ is the input, $b_n$ is the output, and $Q(\cdot)$ is a binary quantizer with the form

$$Q(y) = \begin{cases} 1 & \text{if } y > 0.5 \\ 0 & \text{if } y \leq 0.5 \end{cases} .$$

where we assume that $e_0 = 0$ and the algorithm is run for $n \geq 1$.

Furthermore, define $d_n = x_n - b_n$.

a) Draw a flow diagram for this algorithm. Make sure to label all the signals in the flow diagram using the notation defined above.

b) Calculate $b_n$ for $n = 1$ to 10 when $x_n = 0.25$ and $e_0 = 0$.

c) Calculate an expression for $d_n$ in terms of the quantization error $e_n$.

d) Calculate an expression for $\sum_{n=1}^{N} d_n$ in terms of the quantization error $e_n$.

e) What does the result of part d) above tell you about the output of error diffusion?

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