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Electrical Engineering
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H.W solution #6.

2003. Spring.

1. Spring 2002 Final. Problem 1.

a). $y(m,n) = x(m,n) + h(m,n) * e(m,n)$

$= x(m,n) + \delta(m-1,n) * e(m,n)$

$= x(m,n) + e(m-1,n)$

$= x(m,n) + y(m-1,n) - z(m-1,n)$.

Hence,

$y(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	0.25	0.5	0.75	0	0.25
$n=1$	0.25	0.5	0.75	0	0.25
$n=2$	0.25	0.5	0.75	0	0.25

$z(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	0	0	1	0	0
$n=1$	0	0	1	0	0
$m=2$	0	0	1	0	0

b) By similiar approach.

$y(m,n) = x(m,n) + y(m-1,n-1) - z(m-1,n-1)$.

Hence,

$y(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	0.25	0.25	0.25	0.25	0.25
$n=1$	0.25	0.5	0.5	0.5	0.5
$m=2$	0.25	0.5	0.75	0.75	0.75

$z(m,n)$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	0	0	0	0	0
$m=1$	0	0	0	0	0
$m=2$	0	0	1	1	1

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Electrical Engineering
Yeesoo Han

c) $d(m,n) = z(m,n) - x(m,n)$

$= y(m,n) - e(m,n) - x(m,n)$

$= y(m,n) - x(m,n) - e(m,n)$

$= h(m,n) * e(m,n) - e(m,n)$

$= (h(m,n) - \delta(m,n)) * e(m,n)$

$= -(\delta(m,n) - h(m,n)) * e(m,n)$

d) From c)

$$D(e^{j\mu}, e^{j\nu}) = -(1 - H(e^{j\mu}, e^{j\nu}))E(e^{j\mu}, e^{j\nu})$$

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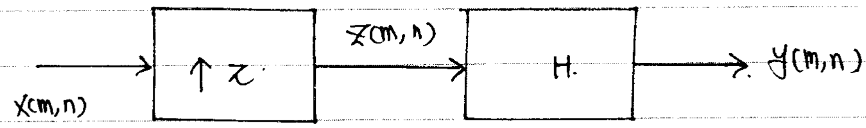
2. Spring 2002 Final Problem 3.

If we plot this system.



a) $z(m,n)$

$$\begin{array}{ccccc} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Hence, $x(m,n)$ becomes.

$$\begin{array}{ccccc} 0 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1 \\ 0 & 0.25 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

b) $h(m,n) = \underbrace{(0.5\delta(m+1) + \delta(m,n) + 0.5\delta(m-1))}_{h_1(m)} \underbrace{(0.5\delta(n+1) + \delta(n) + 0.5\delta(n-1))}_{h_2(n)}$

$H_1(e^{j\mu}) = \text{DTFT}\{h_1(n)\} = 1 + 0.5e^{j\mu} + 0.5e^{-j\mu} = 1 + \cos\mu.$

Hence, $H(e^{j\mu}, e^{j\nu}) = (1+\cos\mu)(1+\cos\nu)$

c) $Y(e^{j\mu}, e^{j\nu}) = Z(e^{j\mu}, e^{j\nu})\, H(e^{j\mu}, e^{j\nu})$

$Z(e^{j\mu}, e^{j\nu}) = X(e^{j2\mu}, e^{j2\nu})$

$\therefore Y(e^{j\mu}, e^{j\nu}) = X(e^{j\mu}, e^{j2\nu})\, H(e^{j\mu}, e^{j\nu})$

d) Advantage : · Easy to compute

· Finite support.

Disadvantage : Not a perfect anti-imaging

3. Spring 2002 Final. Problem 4.

a) $y = \arg\min_{\theta} \sum_{n=0}^{N-1} (X_n - \theta)^2$.

$$\frac{d}{d\theta} y = \sum_{n=0}^{N-1} (X_n - \theta)(-\theta) = 0$$

$$\theta \sum_{n=0}^{N-1} X_n = \theta^2 N.$$

$$\therefore \theta = \frac{1}{N} \sum_{n=0}^{N-1} X_n. \leftarrow \text{sample mean.}$$

b) $y = \arg\min_{\theta} \sum_{n=0}^{N-1} |X_n - \theta|.$

$$\frac{d}{d\theta} y = \sum_{n=0}^{N-1} \text{sign}(X_n - \theta)(-1) = 0$$

$$\sum_{n=0}^{N-1} \text{sign}(X_n - \theta) = 0$$

$$y = \text{median of } \{X_n\}_{n=0}^{N-1}.$$

c). $y = \arg\min_{\theta} \sum_{n=0}^{N-1} |X_n - \theta|^{0.5}$

$$\frac{d}{d\theta} y = \sum_{n=0}^{N-1} 0.5 |X_n - \theta|^{-0.5} \text{sign}(X_n - \theta) = 0$$

$$\therefore \sum_{n=0}^{N-1} \cancel{0.5} (X_n - \theta)^{-0.5} \text{sign}(X_n - \theta) = 0.$$

Hence, $y = \text{root of } \left\{ \sum_{n=0}^{N-1} (X_n - \theta)^{-0.5} \text{sign}(X_n - \theta) \right\}$

d) linear? No!

counter example let $\{X_1^1, X_2^1, X_3^1\} = (0, 1, 0)$

then $y_1 = 0$

and if $\{X_1^2, X_2^2, X_3^2\} = (0, 0, 1)$

then $y_2 = 0$

However $\{X_1^3, X_2^3, X_3^3\} = \{X_1^1, X_2^1, X_3^1\} + \{X_1^2, X_2^2, X_3^2\} = (0, 1, 1)$

then $y_3 = 1$ which is different from $y_1 + y_2 = 0$.

Homogeneous ? yes.

let, $(x_1', x_2', x_3') = \alpha (x_1, x_2, x_3) \quad \forall \alpha > 0, \ \alpha \in \mathbb{R}.$

$$y' = \arg\min_{\theta} \sum_{n=0}^{N-1} |x_n' - \theta|^{0.5}$$

$$= \arg\min_{\theta} \sum_{n=0}^{N-1} |\alpha x_n - \theta|^{0.5}$$

$$= \arg\min_{\theta} \sum_{n=0}^{N-1} \alpha^{0.5} \left| x_n - \frac{\theta}{\alpha} \right|^{0.5}$$

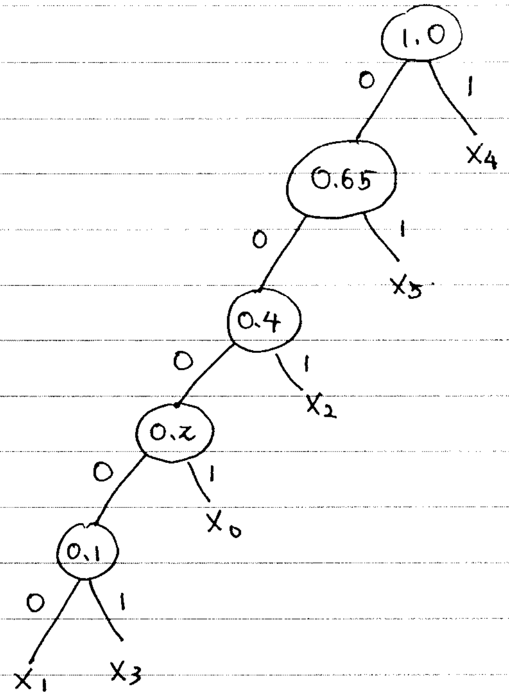
$$= \arg\min_{\theta} \sum_{n=0}^{N-1} \left| x_n - \frac{\theta}{\alpha} \right|^{0.5}$$

$$= \alpha \cdot \arg\min_{\theta} \sum_{n=0}^{N-1} |x_n - \theta'|^{0.5}$$

$$= \alpha \cdot y.$$

4. 2001 Spring Final Problem #1.

a).



b) X_0 : 0001

X_1 : 00000

X_2 : 001

X_3 : 00001

X_4 : 1

X_5 : 01

c) $4\times0.1+5\times0.04+3\times0.2+5\times0.06+1\times0.35+2\times0.25$

$= 2.35$

5. Spring 2001 Final Problem #2.

a) From the orthogonality principle, we have.

$$E[(X-\theta Z)Z^T] = 0$$

$$E[XZ^T] = \theta E[ZZ^T]$$
$$b^T = \theta R.$$

$$\therefore \theta = b^T R^{-1}.$$

b) No! minimum mean square estimator is conditional expectation. This is linear function only X, Z is jointly Gaussian.

c) $$E[|X-\hat{X}|] = E_Z[E[|X-\hat{X}||Z]]$$

$$= E_Z[E[|X-T(Z)||Z]]$$

$$= \int_{\mathbb{R}^M} E[|X-T(Z)||Z]\, p(Z)\, dZ. \quad -(1)$$

Hence, (1) is minimized when $E[|X-T(Z)||Z]$ is minimized.

Hence $T(Z) = \arg\min \int_{\mathbb{R}} |X-T(Z)|\, p(x|Z)\, dx.$

~~This is~~

Hence, $T(Z)$ is median of X given Z.

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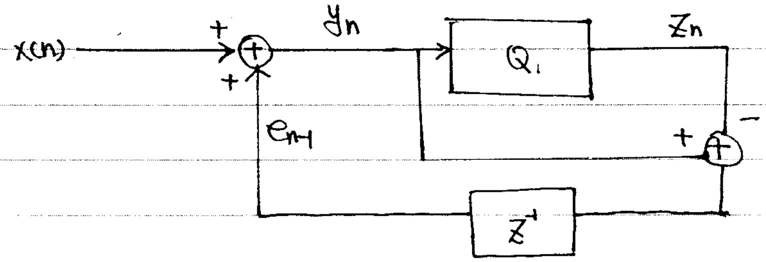
Yeesoo Han

6. Spring 2001 Final Problem #3.

a)



b) $z_n = Q(y_n) = y_m + w_n$.

$e_n = -w_n$.

$y_n = x_n - w_{n-1}$.

$\therefore\ z_n = x_n - w_{n-1} + w_n$.

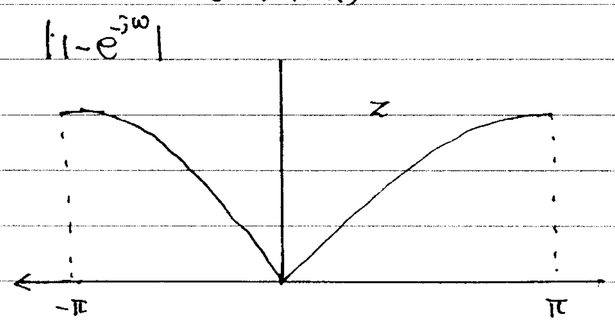
c). $d_n = z_n - x_n$

$= w_n - w_{n-1}$

d). $d_n = w_n - w_{n-1} = w_n * (\delta(n) - \delta(n-1))$.

$D(z) = (1 - z^{-1})\, W(z)$

$= (1 - e^{-j\omega})\, W(z)$



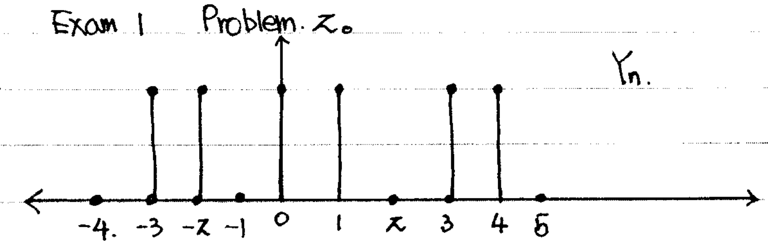
Hence PSD of display error is

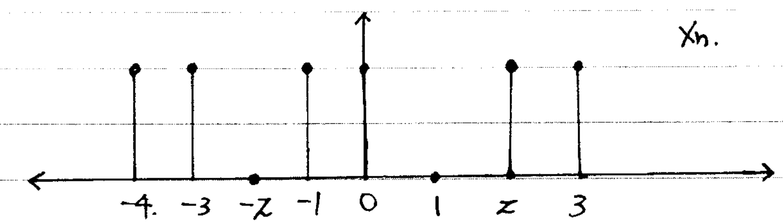
$$|1-e^{-j\omega}|^2 \frac{1}{12\Delta^2}$$

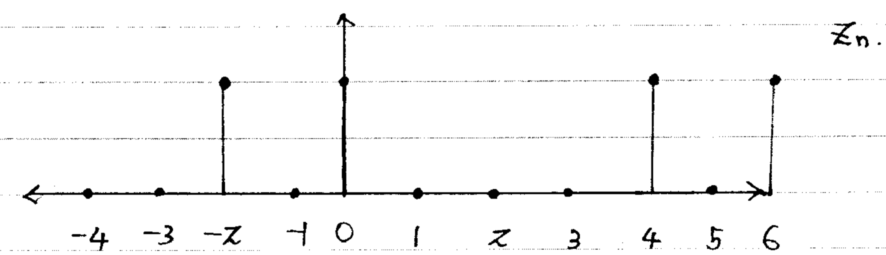
e) HVS acts as lowpass filter. So display error which has high pass filter will be good enough. This kinds of noise power spectrum is called blue noise.

2000 Exam 1 Problem. 2.

a).







b). We know $\theta^* = R_w^{-1} P$

where
$$R_w = \begin{bmatrix} E[Z_{n-1}^2] & E[Z_{n-1} Z_n] & E[Z_{n-1} Z_{n+1}] \\ E[Z_n Z_{n-1}] & E[Z_n^2] & E[Z_n Z_{n+1}] \\ E[Z_{n+1} Z_{n-1}] & E[Z_{n+1} Z_n] & E[Z_{n+1}^2] \end{bmatrix}$$

$$P = \begin{bmatrix} E[Z_{n-1} Y_n] \\ E[Z_n Y_n] \\ E[Z_{n+1} Y_n] \end{bmatrix}$$

c). From the a) we can see $Y_{2k} = Z_{2k}$.

Also note that. $Z_{2k-1} = Z_{2k+1} = 0$

i) when $n = 2\alpha$ (i.e when n is even)

$$\hat{Y}_n = W_n \theta$$
$$= [Z_{n-1} \quad Z_n \quad Z_{n+1}] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$
$$= Z_{n-1}\theta_1 + Z_n \theta_2 + Z_{n+1}\theta_3$$
$$= Z_n \theta_2 \quad (\text{since } Z_{n-1}, Z_{n+1} = 0)$$

$= Z_{2d}\theta_2 = Y_{2d}\theta_2$ (since $Y_{2k} = Z_{2k}$).

$\therefore$ $Y_n = Z_m$ iff $\theta_2 = 1$.

ii) when m is odd.

$$\hat{Y}_n = W_n\theta = [Z_{n-1} \quad Z_n \quad Z_{n+1}]\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}$$

$$= Z_{n-1}\theta_1 + Z_n\theta_2 + Z_{n+1}\theta_3$$

since Z_n is 0, θ_2 can be arbitrary.

From i, ii), we can conclude $\theta_2 = 1$.

d). From c), we can get the value of θ_1, θ_3 when m is odd number and $Z_n = 0$

so, $E[Z_{n-1}Z_n] = E[Z_{n+1}Z_n] = E[Z_n^2] = 0$, $E[Z_{n-1}Z_{n+1}] = \frac{1}{3}$

$E[Z_{n-1}^2] = E[Z_{n+1}^2] = \frac{2}{3}$ (see the picture of Z_n and think when $m \in [0,5]$).

$E[Z_{n-1}Y_n] = \frac{1}{3}$, $E[Z_{n+1}Y_n] = \frac{1}{3}$ (By similar reasoning)

Hence, optimum solution is

$$\begin{bmatrix}\frac{2}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 0\\ \frac{1}{3} & 0 & \frac{2}{3}\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\ 0\\ \frac{1}{3}\end{bmatrix}$$

Hence, $\theta_1 = \theta_2 = \frac{1}{3}$.

8. Fall 1998 Exam #1 Problem #1.

a) From the orthogonality principle,

$$E[Z_n^T(Y_n - Z_n\theta)] = 0.$$

$$\therefore \theta = (E[Z_n^T Z_n])^{-1} E[Z_n^T Y_n]$$

b) From (a), we have.

$$E[Z_n^T Z_n] = \begin{bmatrix} E[X_{n-p}^2] & \cdots & E[X_{n-p}X_{n+p}] \\ \vdots & & \vdots \\ E[X_{n+p}X_{n-p}] & \cdots & E[X_{n+p}^2] \end{bmatrix}$$

since $X_n = Y_n * h_n$

$$R_X(n) = R_Y(n) * h_n * h_{-n} = \sigma^2\delta(n) * h_n * h_{-n} = \sigma^2 h_n * h_{-n}.$$

For $E[Z_n^T \cancel{X_n}]$, from the definition.

$$E[Z_n^T Y_n] = \begin{pmatrix} E[X_{n-p}Y_n] \\ \vdots \\ E[X_n Y_n] \\ \vdots \\ E[X_{n+p}Y_n] \end{pmatrix}$$

$$E[X_{n+k}Y_n] = E[(\sum_{\ell} Y_{n+k-\ell} h_\ell) Y_n]$$

$$= \sum_{\ell} E[Y_{n+k-\ell} Y_n] h_\ell = \sum_{k} \sigma^2 \delta(k-\ell) h_\ell = \sigma^2 h_k.$$

Hence, Define $\rho_k = h_{n+k} * h_{-n}$.

then $\hat{\theta} = (E[Z_n^T Z_n])^{-1} E[Z_n^T Y_n] = \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{2p} \\ \rho_{-1} & \rho_0 & \cdots & \rho_{2p-1} \\ \vdots & & & \\ \rho_{-2p} & & \cdots & \rho_0 \end{bmatrix}^{-1} \begin{bmatrix} h_{-p} \\ \vdots \\ h_0 \\ \vdots \\ h_p \end{bmatrix}$

c)

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c) If we plot overall system

$$Y_n \longrightarrow \boxed{h_n} \xrightarrow{X_n} \boxed{\theta_n} \longrightarrow \hat{Y}_n.$$

$\hat{S}_Y(\omega) = S_X(\omega)|\theta(\omega)|^2$

$S_X(\omega) = S_Y(\omega)\ |H(\omega)|^2$

we want $S_Y(\omega) = \hat{S}_Y(\omega)$

Hence $\hat{S}_Y(\omega) = S_Y(\omega)\ |H(\omega)|^2 |\theta(\omega)|^2$

$$\therefore \quad |\theta(\omega)|^2 = \frac{1}{|H(\omega)|^2}$$

$$|\theta(\omega)| = \frac{1}{|H(\omega)|}.$$

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9. 1997 Exam #2 Problem 1.

a).

0	1	2	3	4	$\boxed{100}$	101	102	103
1	2	2	1	1	$\boxed{3}$	2	2	1

This is output.

10. 1997 Exam # 2 Problem. 2.

a). This is cluster dot screen since dot that turn on are grouped together.

b) we have $T(i,j) = \dfrac{I(i,j)+0.5}{16}$

$h(m,n) = 0.25 > T(i,j)$ if true 1
false 0.

Hence, $h(m,n)$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1997 Exam 2 Problem #3.

a). $$\frac{d}{d\theta_j}E = \frac{d}{d\theta_j}\sum_{i=1}^{N}\rho(y_i-\hat{y}_i)$$

$$= \sum_{i=1}^{N}\rho'(y_i-\hat{y}_i)(-Z_{ij})$$

$$= \sum_{i=1}^{N}-\rho'(y_i-\hat{y}_i)Z_{ij}$$

$$= \sum_{i=1}^{N}-\rho'(y_i-Z_i\theta)Z_{ij}$$

b). $$\theta_j^{(k+1)} = \theta_j^{(k)} + \mu\sum_{i=1}^{N}\rho'(y_i-Z_i\theta)Z_{ij}$$

In vector form.

$$\theta^{(k+1)} = \theta^{(k)} + \mu\sum_{i=1}^{N}\rho'(y_i-Z_i\theta)Z_i^T$$

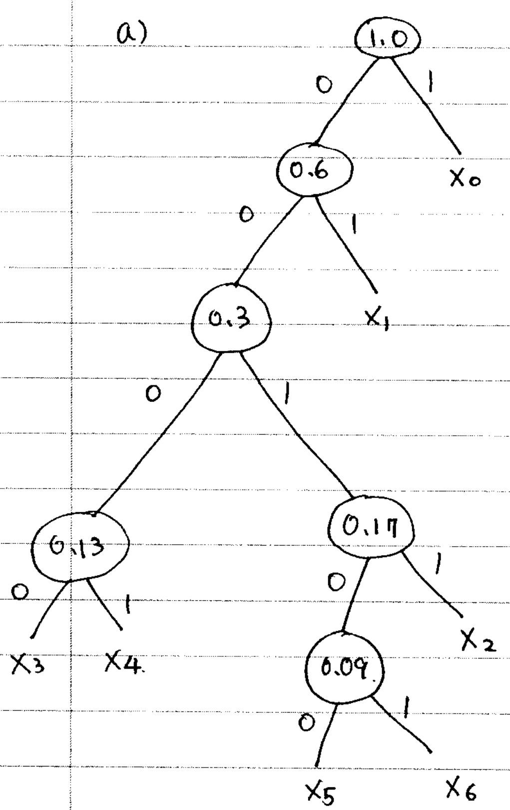
c) $$y_1 - Z_1\theta^{(1)} = 0 - [0 \quad 1 \quad 2 \quad 3 \quad 4]\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} = -2.$$

$$\theta^{(2)} = \theta^{(1)} + \frac{1}{10}\rho'(-2)\begin{bmatrix}0\\1\\2\\3\\4\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} - \frac{1}{10}\begin{bmatrix}0\\1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\-\frac{1}{10}\\0.8\\-0.3\\-0.4\end{bmatrix}$$

1997. Fall Final Problem #4.

a)



b) X_0 : 1

X_1 : 01

X_2 : 0011

X_3 : 0000

X_4 : 0001

X_5 : 00100

X_6 : 00101.

c) $0.4\times1 + 0.3\times2 + 0.08\times4 + 0.07\times4 + 0.06\times4 + 0.05\times5 + 0.04\times5$

$= 2.29.$

d). Coding in blocks.

Code blocks of the sequence

$\underbrace{X_0, \cdots X_{m-1},}_{Y_0} \underbrace{X_m \cdots X_{2m-1}}_{Y_1}$

$Y_n \in \{0, \cdots, 6^m - 1\}$

From the class note,

$$H(X_n) \leq \bar{n}_x < H(X_n) + \frac{1}{m}$$

where $H(X_n) \triangleq -\sum_{i=0}^{6} p_i \log_2 p_i$