EE 637 Final

April 29, Spring 2008

Name: Key

Instructions:

- This is a 120 minute exam containing **five** problems.
- Each problem is worth 40 points for a total score of 200 points
- You may **only** use your brain and a pencil (or pen) to complete this exam.
- You **may not** use your book, notes, or a calculator, or any other electronic or physical device for communicating or accessing stored information.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- CTFT

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-f)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(f) e^{-j2\pi f t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(f/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} x(-f)$$

$$x(t) e^{j2\pi f_0 t} \overset{CTFT}{\Leftrightarrow} X(f-f_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} X(f) * Y(f)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j2\pi f + a)^n}$$

- CSFT

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy \\ f(x,y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv \end{aligned}$$

- DTFT

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Rep and Comb relations

$$\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\mathrm{comb}_T[x(t)] = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$\mathrm{comb}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{rep}_{\frac{1}{T}}[X(f)]$$

$$\mathrm{rep}_T[x(t)] \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \mathrm{comb}_{\frac{1}{T}}[X(f)]$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t-kT) \\ S(e^{j\omega}) &= Y\left(e^{j\omega T}\right) \end{aligned}$$

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Problem 1.(40pt)

Consider a 2D linear space-invariant filter with input $x(m,n)$, output $y(m,n)$, and impulse response $h(m,n)$, so that

$$y(m,n) = h(m,n) * x(m,n) \ .$$

Furthermore, let the impulse response be given by

$$h(m,n) = \begin{cases} \frac{1}{81} & \text{for } |m| \le 4 \text{ and } |n| \le 4 \\ 0 & \text{otherwise} \end{cases} \ .$$

8' a) What is the DC gain of this filter? Justify your answer?

8' b) Assuming that you implement this filter directly with 2D convolution, how many multiplies are required per output value?

8' c) Find a separable decomposition of $h(m,n)$ so that

$$h(m,n) = g(m)\, f(n)$$

where $g(m)$ and $f(n)$ are 1D functions.

8' d) Explain how the functions $g(m)$ and $f(n)$ can be used to compute $y(m,n)$. What is the advantage of this approach? Be specific.

8' e) Calculate a simple expression for $G(e^{j\omega})$ the DTFT of $g(m)$.

a). $H(e^{ju}, e^{jv}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m,n) e^{-jum} e^{jvn} = \frac{1}{81} \sum_{m=-4}^{4} \sum_{n=-4}^{4} e^{-jum} e^{-jvm}$

$\therefore$ the DC gain $H(u=0, v=0) = \frac{1}{81} \sum_{m=-4}^{4} \sum_{n=-4}^{4} 1 = \frac{1}{81} \times 9 \times 9 = 1$.

b). $y(m,n) = h(m,n) * x(m,n)$. 2D convolution.

since $h(m,n)$ is a 9×9 smoothing filter.

$\therefore$ $9\times 9 = 81$ multiplies needed. per output

c). Let $g(m) = \begin{cases} \frac{1}{9} & \text{for } |m| \le 4 \\ 0 & \text{o.w.} \end{cases}$ & $f(n) = \begin{cases} \frac{1}{9} & \text{for } |n| \le 4 \\ 0 & \text{o.w} \end{cases}$

we have $h(m,n) = g(m) f(n)$.

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(d). the filter $h(m,n) = g(m) f(n)$ is separable,
we can first filter the image along the horizontal direction, i.e., each line is filtered by $g(m)$, we get the intermediate image $\tilde{x}(m,n)$; then, each column of $\tilde{x}(m,n)$ is filtered using $f(n)$, we obtain the desired image $y(m,n)$.

advantage: need only $9+9=18$ multiplies
lower complexity compared to 2D convolution

(e). $$G(e^{j\omega}) = \sum_{m=-\infty}^{\infty} g(m) e^{-j\omega m}$$

$$= \frac{1}{9} \sum_{m=-4}^{4} e^{-j\omega m}$$

$$= \frac{1}{9} \frac{e^{j4\omega}(1-e^{-j9\omega})}{1-e^{-j\omega}}$$

$$= \frac{1}{9} \frac{\cancel{e^{-j\frac{1}{2}\omega}}(e^{j\frac{9}{2}\omega} - e^{-j\frac{9}{2}\omega})}{\cancel{e^{-j\frac{1}{2}\omega}}(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega})}$$

$$= \frac{1}{9} \times \frac{\sin\frac{9}{2}\omega}{\sin\frac{1}{2}\omega}$$

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Problem 2.(40pt)

Let X_n be a discrete-time random process with i.i.d. samples, and distribution given by $P\{X_n = k\} = p_k$ where

$$(p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7) = \left(\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right)$$

8' a) What is the value of p_8? Why?

8' b) Calculate $H(X_n)$ in bits.

8' c) Draw the Huffman tree and determine the binary Huffman code for each possible symbol.

8' d) Calculate the expected code length per symbol.

8' e) Are there better codes for X_n? If so, what are they? If not, why not?

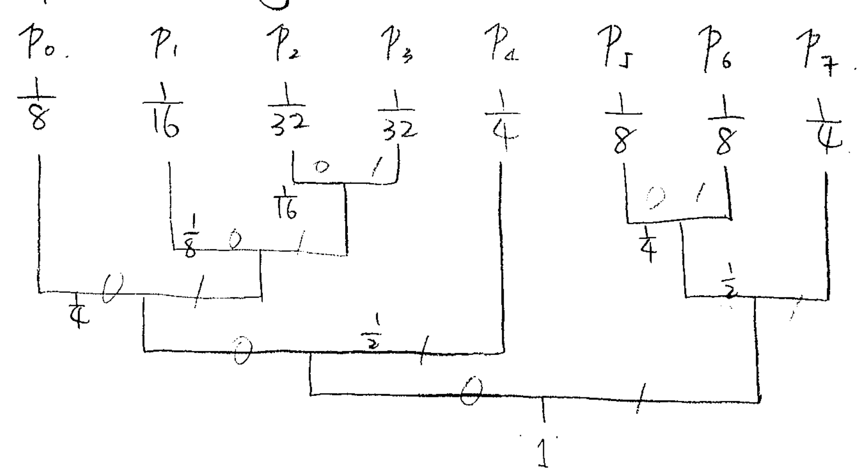
a). Since $\sum_{k=0}^{7} p_k = 1$, if the discrete-time random process X_n takes value $X_n = 8$, the associated probability $P(\{X_n = 8\}) = p_8$ should be "0", i.e.. $p_8 = 0$.

b). $H(X_n) = -\sum_{k=0}^{7} p_k \log_2 p_k$

$= \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{32}\times 5 + \frac{1}{32}\times 5 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 3 + \frac{1}{4}\times 2$

$= \frac{43}{16}$ bits

c) Binary Huffman Coding.



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p_k	Symbol $X_n = k$	Huffman Code	Length l_k (in bits).
1/8	0	000	3
1/16	1	0010	4
1/32	2	00110	5
1/32	3	00111	5
1/4	4	01	2
1/8	5	100	3
1/8	6	101	3
1/4	7	11	2

(d). expected length $= \sum_{k=0}^{7} p_k l_k = \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{32}\times 5 + \frac{1}{32}\times 5$

$+ \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 3 + \frac{1}{4}\times 2$

$= \frac{43}{16}$ bits $= H(X_n)$.

(e). Since the entropy for this source is $H(X_n) = \frac{43}{16}$ bits, which is the lower bound of bitrate to represent the source data, for any coder, we'll have expected length $L \geq H$.

The above Huffman Coder achieves the lower bound since $L = H$.

$\Rightarrow$ No other coder will do better, Huffman Coder is optimal for this source data.

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Problem 3.(40pt)

Let $x(m,n)$ be a gamma corrected image with $\gamma = 2.2$ that takes values in the range 0 to 255 with 0 being perfect black and 255 being the lightest white. Also let $T(m,n)$ be an array of random thresholds which are i.i.d. and are uniformly distributed on the discrete set $\{0, 1, \cdots, 255\}$.

The binary halftone is then computed as

$$b(m,n) = \begin{cases} 255 & \text{if } x(m,n) > T(m,n) \\ 0 & \text{otherwise} \end{cases}$$

$b(m,n) = 255$

Assume that $b(m,n) = 0$ is displayed as perfect black, and $\cancel{b(m,0)}$ is displayed as the lightest white. Also assume the the lightest white has luminance I_o in units proportional to energy.

7' a) What is the relationship between $x(m,n)$ and $I(m,n)$ where $I(m,n)$ is the luminance of the image in units proportional to energy?

7' b) Compute the expected value of $b(m,n)$ as a function of $x(m,n)$.

7' c) Compute the variance of $b(m,n)$ as a function of $x(m,n)$.

7' d) What is the average luminance of the image $b(m,n)$ when $x(m,n) = 127$?

6' e) What should the average luminance of the image be when $x(m,n) = 127$?

6' f) What should be done to more accurately render the luminance using this halftoning method?

a). $X(m,n)$ is the gamma corrected image. $I(m,n)$ is in linear scale

$$\therefore \quad X(m,n) = 255 \left(\frac{I(m,n)}{I_0} \right)^{\frac{1}{\gamma}} \quad \text{where } \gamma = 2.2$$

b). $E[b(m,n)] = 255 \times P(\{b(m,n) = 255\}) + 0 \times P(\{b(m,n) = 0\})$

$= 255 \times P(\{T(m,n) \le X(m,n)\})$

$\because$ $T(m,n)$ iid uniformly distributed on set $\{0, 1, \cdots, 255\}$.

$$\therefore \quad P(\{T(m,n) = i\}) = \frac{1}{256} \qquad i = 0, 1, \cdots, 255.$$

and $P(\{T(m,n) \le X(m,n)\}) = \sum_{i=0}^{X(m,n)} P(\{T(m,n) = i\}) = \frac{X(m,n)+1}{256}$

$$\Rightarrow E[b(m,n)] = \frac{255}{256}(X(m,n) + 1)$$

c). $Var(b(m,n)) = E[b^2(m,n)] - (E[b(m,n)])^2$

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Similarly, $E[b^2(m,n)] = 255^2 \, P(\{b(m,n) = 255\}) = \frac{255^2}{256}(x(m,n)+1)$

$\therefore \mathrm{Var}(b(m,n)) = \frac{255^2}{256}(x(m,n)+1) - \frac{255^2}{256^2}(x(m,n)+1)^2$

$= \frac{255^2}{256}(x(m,n)+1)\left(1 - \frac{x(m,n)+1}{256}\right)$

d). luminance of $b(m,n)$ is $I_0 \frac{b(m,n)}{255}$, if $x(m,n) = 127$

$\therefore E\left[I_0 \frac{b(m,n)}{255}\right] = \frac{I_0}{255} E[b(m,n)] = \frac{I_0}{255} \times \frac{255}{256} \times (127+1) = \frac{1}{2} I_0$

e) the average luminance should be

$I(m,n) = I_0 \left(\frac{x(m,n)}{255}\right)^{\gamma} = I_0 \left(\frac{127}{255}\right)^{\gamma}$

f). we should use

$$b(m,n) = \begin{cases} 255 & \text{if } \tilde{x}(m,n) \geq T(m,n) \\ 0 & \text{o.w.} \end{cases}$$

where $\tilde{x}(m,n)$ is the un-Gamma Corrected image.

i.e.. $\tilde{x}(m,n) = 255\left(\frac{x(m,n)}{255}\right)^{\gamma}$ in the linear scale

in that case, if $x(m,n) = 127$

$E\left[I_0 \frac{b(m,n)}{255}\right] = \frac{I_0}{255} E[b(m,n)] = \frac{I_0}{255} \frac{255}{256}(\tilde{x}(m,n)+1)$

$\approx \frac{I_0}{256} \tilde{x}(m,n) = \frac{I_0}{256} 255 \left(\frac{127}{255}\right)^{\gamma}$

$\approx I_0 \left(\frac{127}{255}\right)^{\gamma}$.

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Problem 4.(40pt)

Consider a discrete-time random process X_n with i.i.d. samples that are Gaussian with mean 0 and variance σ^2.

The rate distortion relation for this source is then given by

$$R(\Delta) = \max\left\{\frac{1}{2}\log_2\left(\frac{\sigma^2}{\Delta^2}\right), 0\right\}$$
$$D(\Delta) = \min\left\{\sigma^2, \Delta^2\right\}$$

8' a) Plot the minimum possible rate (y-axis) versus distortion (x-axis) required to code this source when $\sigma^2 = 1$.

8' b) If we require that the distortion $D \leq \sigma^2$, then what is the minimum (lower bound) on the number of bits per sample that is required to transmit this signal?

8' c) If we require that the distortion $D \leq \frac{\sigma^2}{(32)^2}$, then what is the minimum (lower bound) on the number of bits per sample that is required to transmit this signal?

8' d) How many bits per sample are required in order to achieve zero distortion?

8' e) Describe how you would design a lossy coder for this signal assuming that your objective is to achieve a bit rate of approximately 8 bits per sample.

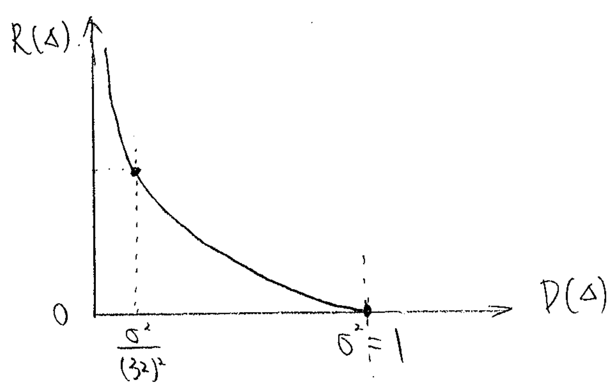
a). $\sigma^2 = 1$, $D(\Delta) = \min\{1, \Delta^2\}$.

if $0 < \Delta < 1$, $D(\Delta) = \Delta^2$, $\frac{\sigma^2}{\Delta^2} = \frac{1}{\Delta^2} > 1 \Rightarrow \frac{1}{2}\log_2\left(\frac{\sigma^2}{\Delta^2}\right) > \frac{1}{2}\log_2 1 = 0$.

$\therefore R(\Delta) = \frac{1}{2}\log_2\left(\frac{1}{\Delta^2}\right) = \frac{1}{2}\log_2\left(\frac{1}{D(\Delta)}\right)$

if $\Delta \geq 1$, $D(\Delta) = \sigma^2 = 1$. $\frac{\sigma^2}{\Delta^2} \leq 1$. $\frac{1}{2}\log_2\left(\frac{\sigma^2}{\Delta^2}\right) < 0 \Rightarrow R(\Delta) = 0$

therefore, the plot is.



b). $D \leq \sigma^2$, the minimum number of bits per sample

$R = 0$ since $\frac{1}{2}\log_2\frac{\sigma^2}{D^2} \geq \frac{1}{2}\log_2\frac{\sigma^2}{\sigma^2} = 0$

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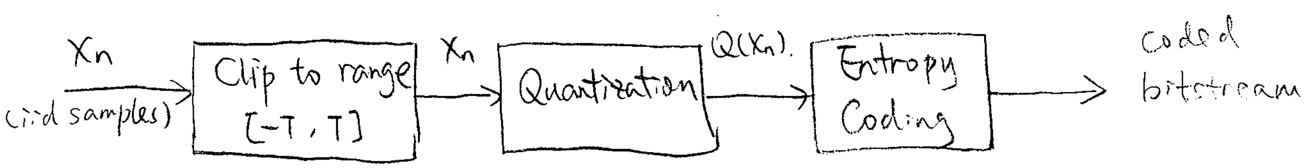
c). when $D \le \frac{\sigma^2}{(32)^2}$ $\quad R = \frac{1}{2}\log_2\left(\frac{\sigma^2}{D^2}\right) \geqslant 5$ (bits).

d). if $D=0$ wanted, $R = \frac{1}{2}\log_2\left(\frac{\sigma^2}{D^2}\right) \longrightarrow \infty$

$\therefore$ infinite number of bits required for zero distortion, which is always impractical. therefore, we generally consider lossy compression, i.e., distortion $\neq 0$.

(e) want $R \approx 8$ bits/sample, then $D \approx \frac{\sigma^2}{(256)^2}$



① We first clip the source samples in the range $[-T, T]$ where $T \gg \sigma$. (for example, take $T=3\sigma$, then about 99% of the samples will be in $[-3\sigma, 3\sigma]$.)

②. Uniform quantization

Since the distortion for a uniform quantization is $\frac{1}{12}\Delta^2$ where Δ is the quantization step. $\Rightarrow D = \frac{1}{12}\Delta^2 = \frac{\sigma^2}{(256)^2}$

$$\Rightarrow \Delta = \sqrt{12}\,\frac{\sigma}{256}.$$

and $Q(X_n) = \text{round}\left(\frac{X_n}{\Delta}\right) = \text{round}\left(\frac{\sqrt{12}\,\sigma}{256}\right)$

③ Entropy Coding the quantized data, and get about 8 bits/sample.

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Problem 5.(40pt)

Consider an MRI that only images in one dimension, x. So for example, the object being imaged might be a thin rod oriented along the x-dimension.

In this example, assume that the magnetic field strength at each location is given by

$$M_o + G(t)x$$

where M_o is the static magnetic field strength and $G(t)x$ is the linear gradient field in the x dimension. Then the frequency of precession for a hydrogen atom (in rad/sec) is given by the product of γ, the gyromagnetic constant, and the magnetic field strength.

8' a) Calculate $\omega(x,t)$, the frequency of precession of a hydrogen atom at location x and time t.

8' b) Calculate $\phi(x,t)$, the phase of precession of a hydrogen atom at location x and time t assuming that $\phi(x,0) = 0$.

8' c) Calculate $r(x,t)$, the signal radiated from hydrogen atoms in the interval $[x, x+dx]$ at time t.

8' d) Calculate $r(t)$, the signal radiated from hydrogen atoms along the entire object.

8' e) Calculate an expression for $a(x)$, the quantity of processing hydrogen atoms along the thin rod, from the function $r(t)$.

a). $w(x,t) = \gamma(M_o + G(t)x)$

b) $$\frac{d\phi(x,t)}{dt} = w(x,t) = \gamma(M_o + G(t)x)$$

$$\Rightarrow \phi(x,t) = \int_0^t \gamma(M_o + G(\tau)x)\,d\tau$$

$$= \int_0^t \gamma M_o\,d\tau + \int_0^t \gamma G(\tau)x\,d\tau$$

$$= \gamma M_o t + x k_x(t)$$

where $k_x(t) = \int_0^t \gamma G(\tau)\,d\tau$

c) $r(x,t) = a(x)e^{j\phi(t)} = a(x)e^{j\gamma M_o t}e^{jk_x(t)x}\,dx$

d) $r(t) = e^{j\gamma M_o t}\int_{-\infty}^{+\infty} a(x)e^{jk_x(t)x}\,dx = e^{j\gamma M_o t}A(-k_x(t))$

where $A(\omega) = \mathcal{F}\{a(x)\}$

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(e). From (d) $r(t) = e^{j\gamma M_0 t} A(-k_x(t))$ where $A(\omega) = \mathcal{F}(a(x))$

$$\Rightarrow a(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{j\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(-\omega) e^{-j\omega x} d\omega$$

i.e., $a(x)$ is obtained by the following steps:

① Adjusting values of $G(t)$ so that

$\omega(t) \triangleq k_x(t) = \int_0^t \gamma G(\tau) d\tau$ goes from $-\omega_{max}$ to $-\omega_{max}$

② record the values of $r(t)$ for different values of $\omega(t)$

③ Compute $A(-\omega(t)) = r(t) e^{-j\gamma M_0 t}$

④ Using the calculated values $A(-\omega(t))$ to compute

$$a(x) = \frac{1}{2\pi} \int_{-\omega_{max}}^{\omega_{max}} A(-\omega) e^{-j\omega x} d\omega.$$

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